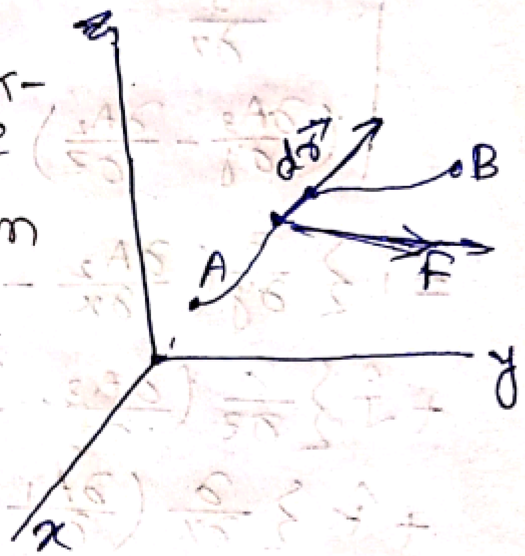
* Vector integration :-

① Line integral :-

Example :- Suppose we need to calculate the work-done by a variable force f in replacing the object from A to B is shown in the figure. The small amount of work dw in displacement $d\vec{r}$ is, $dW = \vec{F} \cdot d\vec{r}$.



$d\vec{r}$ can be written as,

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

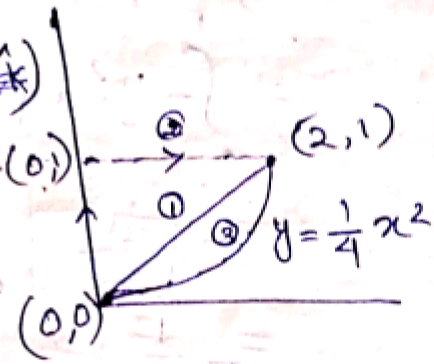
Total workdone will be,

$$\omega = \int_A^B F \cdot d\vec{r} \quad \text{———} ①$$

Although F is a function x, y and z also $d\vec{r}$ is a function of x, y and z with the help of the ~~curve the~~ relation between x, y, z from equation of the curve the integral defined in eqn ① will be converted into an ordinary integral of f^n of one variable. Such integral is called line integral.

eg :—

① If $\vec{F} = xy\,\hat{i} - y^2\,\hat{j}$, find the workdone by $\vec{F}$ along the paths integrated by the figure from (0,0) to (2,1).



Soln. $W = \int \vec{F}.d\vec{r}$

$= \int (xy\,\hat{i} - y^2\,\hat{j})(dx\,\hat{i} + dy\,\hat{j} \cancel{+ dz\,\hat{k}})$

$= \int xy\,dx - y^2\,dy$

equation of the graph,

$$\text{slope}(m) = \cancel{\frac{x_2 - x_1}{y_2 - y_1}}\ \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1 - 0}{2 - 0}$$

$$= \frac{1}{2}$$

$\because \cancel{y}\ y - y_1 = m(x - x_1)$

$\Rightarrow y = mx.$

$= \frac{1}{2}x$

$\therefore dy = \frac{1}{2}dx$

$\therefore W = \int \frac{x^2}{2}dx - \frac{x^2}{8}dx$

$$= \left(\frac{1}{2} - \frac{1}{8}\right)\int_0^2 x^2\,dx$$

$$= \frac{3}{8}\cdot\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{8}{8} = 1. \quad (\text{path } ①)$$

path ② :—

Workdone from (0,0) to (0,1)

$x = 0, \; dx = 0$

$W = \int (xy\,dx - y^2 dy)$

$= \int_0^1 -y^2 dy$

$= -\left[\frac{y^3}{3}\right]_0^1$

$= -\frac{1}{3}$

Work done from (0,1) to (2,1)

$y = 1 \;, dy = 0$

$W = \int (xy\,dx - y^2 dy)$

$= \int_0^2 x\,\cancel{dy}\,dx$

$= \frac{1}{2}\left[x^2\right]_0^2$

$= \frac{4}{2} = 2$

Total Workdone $= -\frac{1}{3} + 2$

$= \frac{5}{3}$

path ③ :—

$W = \int (xy\,dx - y^2 dy)$

$\because y = \frac{1}{4}x^2$

$$dy = \frac{1}{4} 2x \, dx$$
$$= \frac{1}{2} x \, dx$$

$$W = \int \frac{x^3}{4} dx - \frac{x^5}{32} dx$$
$$= \frac{1}{4}\left[\frac{x^4}{4}\right]_0^2 - \frac{1}{32}\left[\frac{x^6}{6}\right]_0^2$$
$$= \frac{1}{\cancel{16}} \cdot \cancel{16} - \frac{1}{\cancel{32} \times \cancel{6}_3} \cdot \cancel{64}$$
$$= 1 - \frac{1}{3}$$
$$= \frac{2}{3}$$

Q.-2. Suppose $\vec{F} = -3x^2 \hat{i} + 5xy \hat{j}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ where c is the curve in xy plane $y = 2x^2$ from (0,0) to (1,2).

Soln. $W = \int_c \vec{F} \cdot d\vec{r}$

$$= \int_c (-3x^2 \hat{i} + 5xy \hat{j}) \cdot (dx \hat{i} + dy \hat{j})$$
$$= \int_c -3x^2 dx + 5xy \, dy$$

$\because y = 2x^2$

$dy = 4x \, dx$

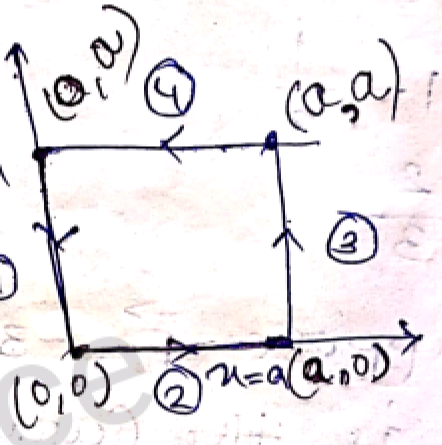
$$W = \int_0^1 -3x^2 dx + 5x \cdot 2x^2 \cdot 4x \, dx$$
$$= \int_0^1 -3x^2 dx + 40x^4 dx$$
$$= -3\left[\frac{x^3}{3}\right]_0^1 + 40\left[\frac{x^5}{5}\right]_0^1$$

$= -1 + 8$

$= 7$

Q3: – Evaluate $\int \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and c is the boundary of the square in the plane $z=0$ and boundary by the lines $z=0$, $x=0$, $y=0$, $x=a$, $y=a$.

Soln. $W = \int \vec{F}\cdot d\vec{r}$



$= \int (x^2\hat{i} + xy\hat{j})(dx\hat{i} + dy\hat{j})$

$= \int_c x^2dx + xydy$

along path ①,

$x=0,\ dx=0$

$W = \int_0^a 0.ydy$

$= 0$

along path ②,

$y=0,\ dy=0,\ x=a$

$W = \int_0^a x^2 dx$

$= \left[\frac{x^3}{3}\right]_0^a$

$= \frac{a^3}{3}$

along path 3,

$$x=a, \quad dx=0$$

$$W=\int_0^a ay\,dy$$

$$=a\left[\frac{y^2}{2}\right]_0^a$$

$$=a\,\frac{a^2}{2}$$

$$=\frac{a^3}{2}$$

along path 4,

$$y=a, \quad dy=0$$

$$=\int_a^0 x^2\,dx$$

$$=\left[\frac{x^3}{3}\right]_a^0$$

$$=-\frac{a^3}{3}$$

$$\text{Total}=0+\frac{a^3}{3}+\frac{a^2}{2}-\frac{a^3}{3}$$

$$=\frac{a^3}{2}$$

Q4:— Suppose $A=(3x^2+6y)\hat{i}-14yz\hat{j}+20xz^2\hat{k}$ evaluate $\int_c A.d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along following paths C. (a) $x=t$, $y=t^2$, $z=t^3$.

Soln. $W=\int A.d\vec{r}=\int(3x^2+6y)\,dx-14yz\,dy+20xz^2\,dz$

$\because x = t, \quad y = t^2, \quad z = t^3$

$\therefore dx = dt, \quad dy = 2t\,dt, \quad dz = 3t^2 dt$

$$= \int_0^1 (3t^2 + 6t^2)\,dt - \cancel{14t}\, 28t^6 dt + 60 t^9 dt$$

$$= 9\left[\frac{t^3}{3}\right]_0^1 - 28\left[\frac{t^7}{7}\right]_0^1 + 60\left[\frac{t^{10}}{10}\right]_0^1$$

$= 3 - 4 + 6$

$= 5$ //

(b) the straight line from (0,0,0) to (1,0,0) then to (1,1,0) and then to (1,1,1).

Sol^n. path ① :— from (0,0,0) to (1,0,0)

$y = 0, \quad dy = 0$

$z = 0, \quad dz = 0$

$$A \cdot d\vec{r} = \int (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2 dz$$

$$= \int_0^1 (3x^2 \cancel{+ 6y})\,dx$$

$$= 3\left[\frac{x^3}{3}\right]_0^1$$

$= \cancel{3}\, 1$ //

path 2 :— from (1,0,0) to (1,1,0)

$z = 0, \quad dz = 0$

$x = 1, \quad dx = 0$

$$A.d\vec{r} = \int_0^1 -14y.0.dy$$

$$= 0$$

path ③ :- from $(1,1,0)$ to $(1,0,0)$

$x=1$, $dx=0$

$y=1$, $dy=0$

$$A.d\vec{r} = \int_0^1 20xz^2dz$$

$$= 20\left[\frac{z^3}{3}\right]_0^1$$

$$= \frac{20}{3}$$

Total path $= 1 + \frac{20}{3}$

$$= \frac{23}{3}$$

ⓑ the straight line $(0,0,0)$ to $(1,1,1)$.

Soln. Let, $x=t$, $y=t$, $z=t$

$\therefore t=0$ to $t=1$

$$A.d\vec{r} = \int (3t^2+6t)dt - 14t^2dt + 20t^3dt$$

$$= \left[3\frac{t^3}{3} + 6\frac{t^2}{2} - 14\frac{t^3}{3} + 20\frac{t^4}{4}\right]_0^1$$

$$= 1+3-\frac{14}{3}+5$$

$$= 9-\frac{14}{3}$$

$$= \frac{13}{3} //$$

Q 34:- Find $\int \vec{F}\cdot d\vec{r}$ where $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ and c is the path defined by $x = 2t$, $y = t$, $z = t^3$ from $t=0$ to $t=1$ (33)

Soln. $F.d\vec{r} = \int (2y+3)dx + xz\,dy + (yz-x)dz$

$x = 2t$, $dx = 2dt$

$y = t$, $dy = dt$

$z = t^3$, $dz = 3t^2 dt$

$\therefore F.d\vec{r} = \int_0^1 (2t+3)2dt + 2t^4 dt + (t^4 - 2t)3t^2 dt$

$= \left[4\frac{t^2}{2} + 6t + 2\frac{t^5}{5} + 3\frac{t^7}{7} - 6\frac{t^4}{4}\right]_0^1$

$= 2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2}$

$= 8 + \frac{28+30-105}{70}$

$= \frac{560+58-105}{70}$

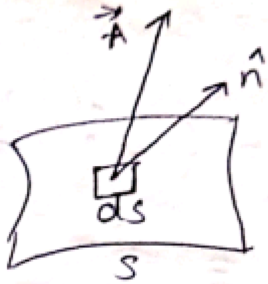
$= \frac{618-105}{70}$

$= \frac{513}{70}$

$= 7.328$ //

* Surface integral :—

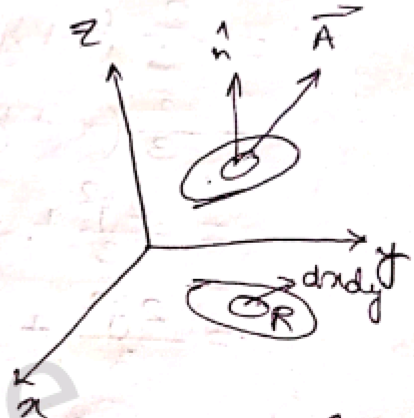
deffⁿ :– The Surface integral of $\vec{A}$ over the surface s is defined as the integral component of A along the normal to the surface. And it is denoted by $\iint_s \vec{A}\cdot\hat{n}\,ds$.



If the surface 's' has a projection R in xy plane then, $ds\cos\theta = dx\,dy$

$$\Rightarrow ds = \frac{dx\,dy}{\cos\theta}$$

$$= \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$



$\hat{n}\cdot\hat{k} = |\hat{n}|\,|\hat{k}|\cos\theta$

$\hat{n}\cdot\hat{k} = \cos\theta$

Now, the surface integral can be written as,

$$= \iint_s \vec{A}\cdot\hat{n}\,ds$$

$$= \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

eg:— ① Evaluate $\iint_s \vec{A}\cdot\hat{n}\,ds$, where $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and s is the surface of the plane $2x+y+2z=6$ in first octant.

Solⁿ. $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+y+2z-6)$

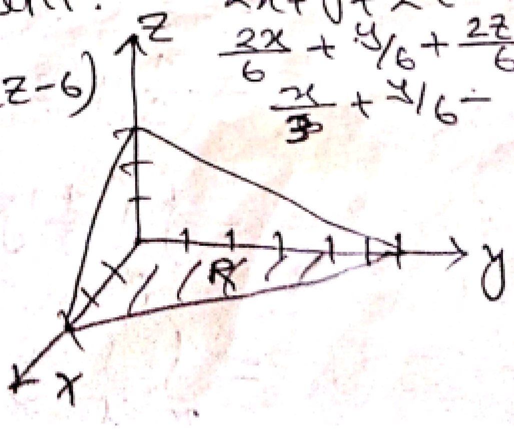
$$= 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\hat{n} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{9}}$$

$2x+y+2z=6$

$\frac{2x}{6} + \frac{y}{6} + \frac{2z}{6} = 1$

$\frac{x}{3} + \frac{y}{6} + \frac{z}{3} = 1$



$$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\hat{n}\cdot\hat{k} = \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2\hat{k}}{3}\right)\cdot\hat{k}$$

$$= \frac{2}{3}$$

$$\vec{A}\cdot\hat{n} = \left\{(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right\}\left\{\frac{2}{3}\hat{i} + \frac{\hat{j}}{3} + \frac{2\hat{k}}{3}\right\}$$

$$= \frac{2}{3}(x+y^2) - \frac{1}{3}2x + \frac{2}{3}\cdot 2yz$$

$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2y^2}{3} + \frac{4}{3}yz$$

On the plane
$2x+y+2z=6$
$\Rightarrow 2z = 6-2x-y$
$\therefore z = \frac{6-2x-y}{2}$

$$= \frac{2y^2}{3} + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$$= \frac{2y^2}{3} + \frac{24y-8xy-4y^2}{6}$$

$$= \frac{4y^2+24y-8xy-4y^2}{6}$$

$$= \frac{24y-8xy}{6}$$

$$= \frac{8(3y-xy)}{6}$$

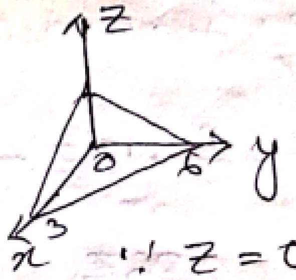
$$= \frac{4}{3}(3y-xy)$$

$$= \frac{4}{3}y(3-x)$$

$$= \iint_R \frac{4}{3}y(3-x)\,\frac{dx\,dy}{2/3}$$

$$= \int_x\int_y 2y(3-x)\,dx\,dy$$

$$= \int_{x=0}^{3} \int_{y=0}^{6-2x} 2y\,(3-x)\,dx\,dy$$



$\because z = 0$

$2x + y = 6$

$\Rightarrow y = 6 - 2x$

$$= \int_{x=0}^{3} (3-x)\,.\,(6-2x)^2\,dx$$

$$= 4\int_{0}^{3} (3-x)^3\,dx$$

Let, $3 - x = t$

$\therefore -dx = dt$

$$= 4\int_{3}^{0} t^3\,dt$$

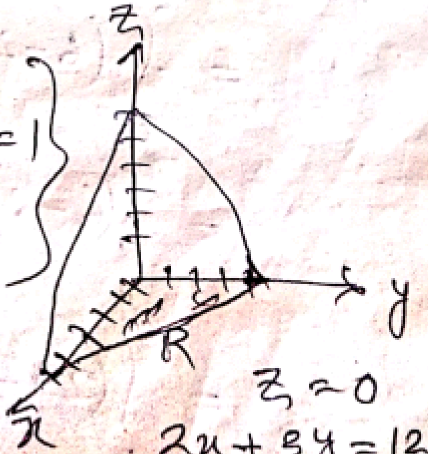
$$= \frac{4}{4}\left[t^4\right]_3^0$$

$$= 3^4 = 81$$

② Evalute $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}$ where s is the plane, $2x + 3y + 6z = 12$ in the first octant.

Solⁿ.

$$\left\{\begin{aligned} &2x + 3y + 6z = 12 \\ &\Rightarrow \frac{2}{12}x + \frac{3}{12}y + \frac{6}{12}z = 1 \\ &\Rightarrow \frac{x}{6} + \frac{y}{4} + \frac{z}{6} = 1 \end{aligned}\right\}$$



$\nabla\varphi = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$$\hat{n} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{49}}$$

$z = 0$

$2x + 3y = 12$

$\cancel{\Rightarrow \frac{x}{6} + \frac{y}{4}}$

$\Rightarrow y = \frac{12 - 2x}{3}$

$$= \frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} + \frac{6\hat{k}}{7} \quad , \quad \hat{k} = 6/7$$

$$\vec{A}\cdot\hat{n} = \frac{2}{7}\cdot 18z - \frac{3}{7}\cdot 12 + \frac{6}{7}\cdot 3y$$

$$= \frac{36z}{7} - \frac{36}{7} + \frac{18y}{7}$$

$$2x+3y+6z=12$$
$$\Rightarrow z = \frac{12-2x-3y}{6}$$

$$= \frac{36z - 36 + 18y}{7}$$

$$= \frac{\overset{6}{\cancel{36}}\left(\frac{12-2x-3y}{\cancel{6}}\right) - 36 + 18y}{7}$$

$$= \frac{72 - 12x - 18y - 36 + 18y}{7}$$

$$= \frac{36-12x}{7}$$

$$\therefore \; = \iint_R \frac{36-12x}{7} \cdot \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$= \int_{x=0}^{6}\int_{y=0}^{1/3(12-2x)} \cancel{2}(6-2x)\,dx\,dy$$

$$= \int_0^6 (6-2x)\left(\frac{1}{3}(12-2x)\right)dx$$

$$= \int_0^6 (6-2x)\left(4-\frac{2}{3}x\right)dx$$

$$= \int_0^6 \left(24 - 4x - 8x + \frac{4}{3}x^2\right)dx$$

$$= 24[x]_0^6 - 12\left[\frac{x^2}{2}\right]_0^6 - \frac{4}{3}\left[\frac{x^3}{3}\right]_0^6$$

$= 144 - 216 + 96$

$= 24$ //

* Volume Integral :—

If F be a vector and v is the volume enclosed by closed surface then the volume integral is defined as,

$$\iiint_v \vec{F} dv .$$

Examples :—

① If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$ evaluate $\iiint_v \vec{F}.dv$ where v is a region bounded by the surface, $x=0$ $y=0$, $x=2$, $y=4$, $z=x^2$, $z=2$.

Soln. $\iiint_v \vec{F} dv = \iiint_{xyz} \vec{F}\, dx\,dy\,dz$

$$= \int_0^2\int_0^4\int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\, dx\,dy\,dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\, dz$$

$$= \int_0^2 dx \int_0^4 dy \left[z^2\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy \left[(2-x^2)^2\hat{i} - x(2-x^2)\hat{j} + y(2-x^2)\hat{k}\right]$$

$$= \int_0^2 dx \int_0^4 dy\, (2-4x^2+x^4)\hat{i} - (2x-x^3)\hat{j} + (2y-yx^2)\hat{k}$$

* Green's theorem :—

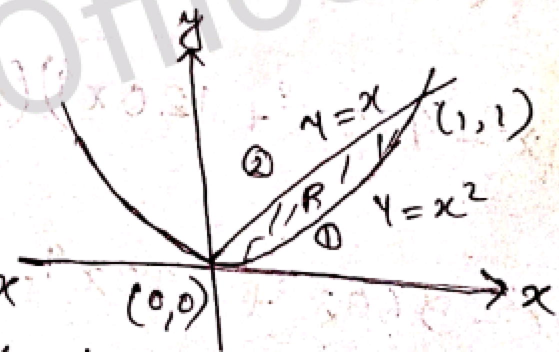
If $\varphi(x,y)$, $\psi(x,y)$, $\frac{\partial \varphi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ are continuous functions over a region R in xy plane bounded by a closed curve C then,

$$\oint_C \varphi dx + \psi dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \varphi}{\partial y}\right) dx\, dy$$

Questions :—

① Verify green's thm in the plane for, $\oint_C (xy+y^2)dx + x^2 dy$, where c is the closed curve bounded by $y=x$ and $y=x^2$

Soln! path ①,



$y=x^2$, $dy=2x dx$

$$= \int_0^1 (x.x^2+x^4)dx + x^2.2x dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1 + 2\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{2}$$

$$= \frac{5+4+10}{20} = \frac{19}{20}$$

path ②, $y=x$, $dy=dx$

$$= \int_1^0 (x^2+x^2)dx + x^2 dx$$

$$= 2\left[\frac{x^3}{3}\right]_1^0 + \left[\frac{x^3}{3}\right]_1^0$$

$$= \left(-\frac{2}{3}\right) + \left(-\frac{1}{3}\right)$$

$$= \frac{-2-1}{3}$$

$$= -\frac{3}{3} = -1$$

Total path,

$$① + ② \Rightarrow \frac{19}{20} - 1$$

$$= \frac{19-20}{20}$$

$$= -\frac{1}{20}$$

$$\because \oint P\,dx + Q\,dy = \iint -\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$\therefore \quad P = xy + y^2$$

$$Q = x^2$$

$$\frac{\partial P}{\partial y} = x + 2y$$

$$\frac{\partial Q}{\partial x} = 2x$$

$$\therefore \oint_c (2x - x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dx\,dy$$

$$= \int_0^1 x\,[y]_{x^2}^{x} - [y^2]_{x^2}^{x}\, dx$$

$$= \int_0^1 x(x-x^2) - (x^2-x^4)\, dx$$

$$= \int_0^1 (x^2 - x^3 - x^2 + x^4)\, dx$$

$$= -\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1$$

$$= -\frac{1}{4} + \frac{1}{5}$$

$$= \frac{-5+4}{20} = -\frac{1}{20}$$

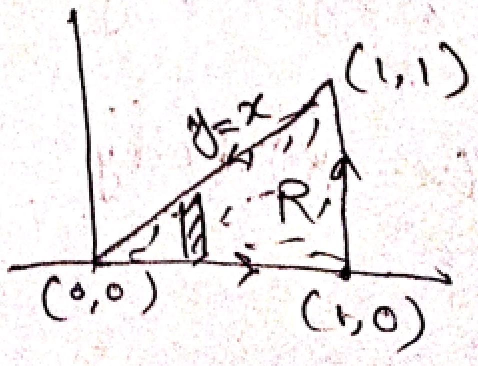
$$\therefore \oint_c P\,dx + Q\,dy = \iint -\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\,dy = -\frac{1}{20}$$

Hence Green's th^m is verify.

② Evaluate $\int x^2 y\,dx + x^2 dy$ where c is the boundary described counter clockwise of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$.

<u>Sol^n.</u> $\oint_c P\,dx + Q\,dy = \iint_R \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\,dy$

Here, $\oint_c x^2 y\,dx + x^2 dy$



$\therefore\ P = x^2 y$

$Q = x^2$

$$\frac{\partial P}{\partial y} = x^2 \quad , \quad \frac{\partial Q}{\partial x} = 2x$$

$$= \iint_R (-x^2 + 2x)\, dx\, dy$$

$$= \int_0^1 \int_0^x (2x - x^2)\, dx\, dy$$

$$= \int_0^1 (2x - x^2)\, x\, dx$$

$$= \int_0^1 2x^2 - x^3\, dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12} = \frac{5}{12}$$

③ Express Green's theorem in vector notation.

Soln. Let, us consider,

$$\vec{A} = P\hat{i} + Q\hat{j}$$

We know;

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\oint P dx + Q dy = \oint (P\hat{i} + Q\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= \oint \vec{A} \cdot d\vec{r}$$

Let. us consider,

$$\nabla \times \vec{A} =$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= \hat{i}\left(-\frac{\partial Q}{\partial z}\right) + \hat{j}\left(\frac{\partial P}{\partial z}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= -\frac{\partial Q}{\partial z}\hat{i} + \frac{\partial P}{\partial z}\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$(\nabla \times \vec{A})\cdot\hat{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\because \oint_c P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\Rightarrow \oint_c \vec{A}\cdot d\vec{r} = \iint (\nabla \times \vec{A})\cdot\hat{k}\;dx\,dy.$$

We can generalized this to any surface s, having a boundary c as,

$$\oint \vec{A}\cdot d\vec{r} = \iint_s (\nabla \times \vec{A})\,\hat{n}\;ds.$$

This is nothing but stoke's theorem.

(4) Evaluate by stoke's theorem $\oint yz\,dx + zx\,dy + xy\,dz$. Where c is the curve $x^2+y^2=1$, $z=y^2$.

Soln. $\oint_c \vec{A}\cdot d\vec{r} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$\vec{A} = yz\hat{i} + zx\hat{j} + xy\hat{k}.$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= 0$$

$$\therefore \oint \vec{A} \cdot d\vec{r} = \iint (\nabla \times \vec{A}) \hat{n}\, ds = 0$$

(5) Using stoke's th^m prove that,

$$\oint \vec{r} \cdot d\vec{r} = 0, \text{ where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Sol^n. $\oint \vec{r} \cdot d\vec{r} = \iint_S (\nabla \times \vec{r}) \hat{n}\, ds$

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= 0$$

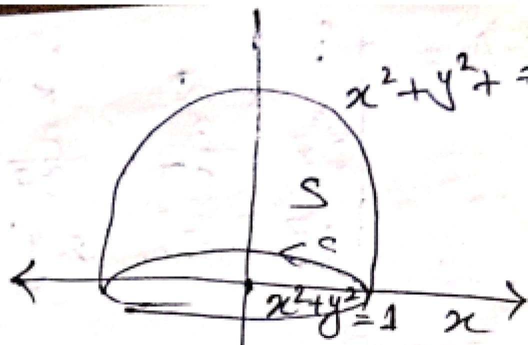
$$\therefore \oint_C \vec{r} \cdot d\vec{r} = 0.$$

(6) Using stoke's theorem calculate $\int_C (2x-y)dx - yz^2 dy - y^2 z\, dz$. Where c is the circle $x^2+y^2=1$. Corresponding to surface of the sphere ~~of~~ unit radius.

Sol^n. $\int_C (2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}\ (dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$A = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & (-y^2z) \end{vmatrix}$$



(45)

$$= \hat{i}(-2yz+2zy) + \hat{j}(0-0) + \hat{k}(0+1)$$

$$= \hat{k}$$

$$= \oint \vec{A} \cdot d\vec{r}$$

$$= \iint (\nabla \times \vec{A}) \hat{n}\, ds$$

$$= \iint \hat{k} \cdot \hat{n}\, ds.$$

$$= \iint \hat{k}\, \hat{n} \frac{dx\, dy}{\hat{n}\, \hat{k}}$$

$$= \iint dx\, dy$$

$$= \pi$$

Other method,

$$= \int_0^1 \int_0^{2\pi} r\, dr\, d\theta$$

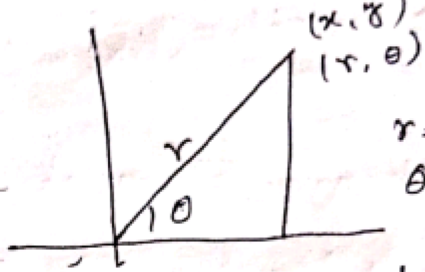
$$= \frac{1}{2} \times 2\pi$$

$$= \pi$$

* Curvilinear Coordinates :-

$x = x(\cancel{r}r, \theta)$
$y = y(r, \theta)$ } transformation eqⁿ

polar co-ord



$r = r(x, y)$
$\theta = \theta(x, y)$

$x = r\cos\theta \Rightarrow r^2 = x^2 + y^2$
$y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}$

defⁿ:- Suppose the rectangular coordinates (x, y, z) of any point in space can be expressed as functions of (u_1, u_2, u_3). Such that

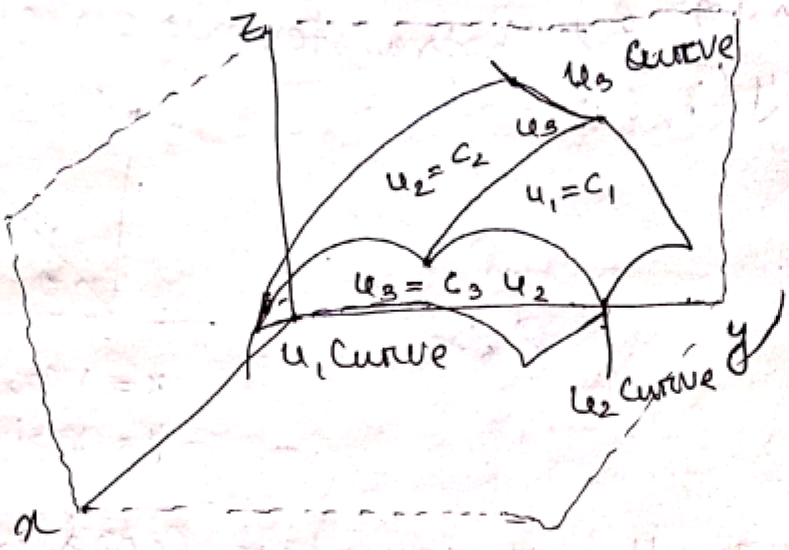
$$\left.\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3) \end{aligned}\right] \text{———} (1)$$

and also (u_1, u_2, u_3) can be expressed in terms of (x, y, z) as

$$\left.\begin{aligned} u_1 &= u_1(x, y, z) \\ u_2 &= u_2(x, y, z) \\ u_3 &= u_3(x, y, z) \end{aligned}\right] \text{———} (2)$$

then the sets of coordinants (u_1, u_2, u_3) is called curvilinear coordinates and eqⁿ (1) and are called transformation equations.

* Orthogonal Curvilinear Coordinates :—



If the Coordinate Surfaces $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ intersect at right angles then the Curvilinear Coordinate is called Othogonal Curvilinear Co-ordinate system.

The position vector of any point in Curvilinear system is represented by $\vec{r} = r(u_1, u_2, u_3)$. The unit vector $\hat{e}_1$ in the direction of u_1 curve is defined as,

$$\hat{e}_1 = \frac{\partial r/\partial u_1}{|\partial r/\partial u_1|}$$

Similarly along with u_2 and u_3 curve the unit vectors are defined as,

$$\hat{e}_2 = \frac{\partial r/\partial u_2}{|\partial r/\partial u_2|}$$

$$\hat{e}_3 = \frac{\partial r/\partial u_3}{|\partial r/\partial u_3|}$$

unit vector can also be define as,

$$\hat{E}_1 = \frac{\nabla u_1}{|\nabla u_1|}$$

$$\hat{E}_2 = \frac{\nabla u_2}{|\nabla u_2|}$$

$$\hat{E}_3 = \frac{\nabla u_3}{|\nabla u_3|}$$

Any vector in Curvilinear coordinate system can be defined as,

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$

Scale factors are defined as,

$$h_1 = \left|\frac{\partial r}{\partial u_1}\right|$$

$$h_2 = \left|\frac{\partial r}{\partial u_2}\right|$$

$$h_3 = \left|\frac{\partial r}{\partial u_3}\right|$$

Therefore, $\frac{\partial r}{\partial u_1} = h_1\hat{e}_1$

$$\frac{\partial r}{\partial u_2} = h_2\hat{e}_2$$

$$\frac{\partial r}{\partial u_3} = h_3\hat{e}_3$$

* Arc length in Curvilinear coordinate system :-

$$dr = \frac{\partial r}{\partial u_1}du_1 + \frac{\partial r}{\partial u_2}du_2 + \frac{\partial r}{\partial u_3}du_3$$

$$= h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_3\hat{e}_3 du_3$$

$$= h_1 du_1 \hat{e}_1 + h_2 du_2 h \hat{e}_2 + h_3 du_3 \hat{e}_3$$

Arc length $= \sqrt{h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2}$

* Volume element in curvilinear coordinate system :-

$$dv = h_1 du_1 h_2 du_2 h_3 du_3 (\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3)$$
$$= h_1 du_1 h_2 du_2 h_3 du_3$$

* Cylindrical Coordinate System :-

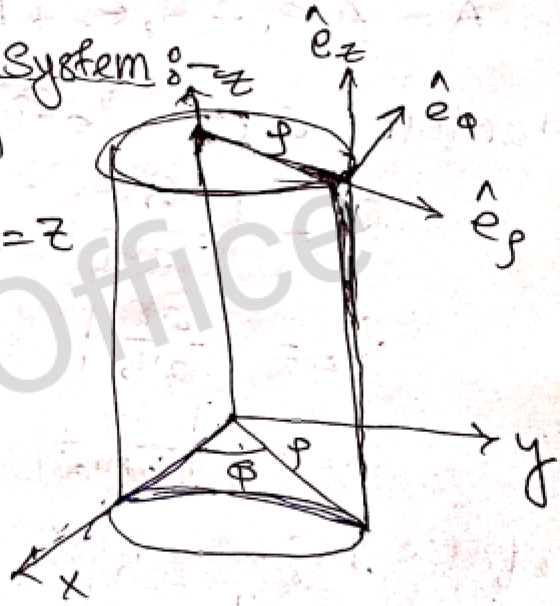
In cylindrical system

$$x = \rho \cos\varphi,\ y = \rho \sin\varphi,\ z = z \quad \text{—①}$$

Where, $\rho \geq 0$

$0 \leq \varphi \leq 2\pi$

$-\infty < z < \infty$



① is transformation equation, unit vectors are,

$$\hat{e}_\rho, \hat{e}_\varphi, \hat{e}_z$$

Examples

① Determine the transformation from cylindrical to rectangular co-ordinates $(x = x(u_1, u_2, u_3) \rightarrow u_1(x, y, z))$

~~Sol^n :-~~ In cylindrical coordinates.

$$y = \rho \sin\varphi,\ x = \rho\cos\varphi,\ z = z$$

$\therefore x^2+y^2=\rho^2$

$\Rightarrow \rho=\sqrt{x^2+y^2}$

and, $\tan\varphi = \frac{y}{x}$

$\Rightarrow \varphi=\tan^{-1}\frac{y}{x}$

$z=z$

② Show that cylindrical coordinate system is orthogonal.

Soln. Since, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$\Rightarrow \vec{r}=\rho\cos\varphi\,\hat{i}+\rho\sin\varphi\,\hat{j}+z\hat{k}$

$$\hat{e}_\rho = \frac{\partial r/\partial\rho}{\left|\frac{\partial r}{\partial\rho}\right|} = \frac{\cos\varphi\,\hat{i}+\sin\varphi\,\hat{j}}{\sqrt{\cos^2\varphi+\sin^2\varphi}} \qquad |h_1=1|$$

$$=\cos\varphi\,\hat{i}+\sin\varphi\,\hat{j}$$

$$\hat{e}_\varphi = \frac{\partial r/\partial\varphi}{|\partial r/\partial\varphi|} = \frac{-\rho\sin\varphi\,\hat{i}+\rho\cos\varphi\,\hat{j}}{\sqrt{\rho^2\sin^2\varphi+\rho^2\cos^2\varphi}}$$

$$=\frac{-\rho(\sin\varphi\,\hat{i}+\cos\varphi\,\hat{j})}{\rho}$$

$$=-\sin\varphi\,\hat{i}+\cos\varphi\,\hat{j}$$

$\therefore h_2=\rho$

$$\hat{e}_z = \frac{\partial r/\partial z}{\partial r/\partial\varphi} = \hat{k}$$

$\therefore h_3=1$

$\hat{e}_\rho \cdot \hat{e}_\phi = -\cos\phi \sin\phi + \sin\phi \cos\phi$

$= 0$

$\hat{e}_\rho \cdot \hat{e}_z = 0$

$\hat{e}_z \cdot \hat{e}_\phi = 0$

Therefore Cylindrical co-ordinate system is othogonal.

③ Represent vector, $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in Cylindrical Co-ordinates.

Solⁿ. Since.

$\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$ ——— ①

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$ ——— ②

$\hat{e}_z = \hat{k}$

① $\times \cos\phi \Rightarrow \cos\phi\,\hat{e}_\rho = \cos^2\phi\,\hat{i} + \sin\phi\cos\phi\,\hat{j}$

② $\times \sin\phi \Rightarrow \sin\phi\,\hat{e}_\phi = -\sin^2\phi\,\hat{i} + \sin\phi\cos\phi\,\hat{j}$

$\Rightarrow \cos\phi\,\hat{e}_\rho \mp \sin\phi\,\hat{e}_\phi = \cos^2\phi\,\hat{i} + \sin\phi\cos\phi\,\hat{j} + \sin^2\phi\,\hat{i} - \sin\phi\cos\phi\,\hat{j}$

$= \hat{i}$

① $\times \sin\phi \Rightarrow \sin\phi\,\hat{e}_\rho = \sin\phi\cos\phi\,\hat{i} + \sin^2\phi\,\hat{j}$

② $\times \cos\phi \Rightarrow \cos\phi\,\hat{e}_\phi = -\sin\phi\cos\phi\,\hat{i} + \cos^2\phi\,\hat{j}$

$\Rightarrow \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi = \sin\phi\cos\phi\,\hat{i} - \sin\phi\cos\phi\,\hat{i} + \sin^2\phi\,\hat{j} + \cos^2\phi\,\hat{j}$

$= \hat{j}$

Given, $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ (52)

$$= z(\cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi) - 2(\rho\cos\varphi)(\sin\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi) + (\rho\sin\varphi)\,\hat{e}_z$$

$$= z\cos\varphi\,\hat{e}_\rho - z\sin\varphi\,\hat{e}_\varphi - 2\rho\cos\varphi\sin\varphi\,\hat{e}_\rho + 2\rho\cos^2\varphi\,\hat{e}_\varphi + \rho\sin\varphi\,\hat{e}_z$$

$$= (z\cos\varphi - 2\rho\cos\varphi\sin\varphi)\,\hat{e}_\rho - (z\sin\varphi - 2\rho\cos^2\varphi)\,\hat{e}_\varphi + \rho\sin\varphi\,\hat{e}_z$$

(4) Express velocity and acceleration in cylindrical Coordinate system.

Soln: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow r = \rho\cos\varphi(\cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi) + \rho\sin\varphi(\sin\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi) + z\,\hat{e}_z$$

$$= \rho\cos^2\varphi\,\hat{e}_\rho - \rho\cos\varphi\sin\varphi\,\hat{e}_\varphi + \rho\sin^2\varphi\,\hat{e}_\rho + \rho\sin\varphi\cos\varphi\,\hat{e}_\varphi + z\,\hat{e}_z$$

$$= \rho(\cos^2\varphi + \sin^2\varphi)\,\hat{e}_\rho + z\,\hat{e}_z$$

$$= \rho\,\hat{e}_\rho + z\,\hat{e}_z$$

$$\frac{dr}{dt} = \frac{d}{dt}(\rho\,\hat{e}_\rho + z\,\hat{e}_z)$$

$$= \dot{\rho}\,\hat{e}_\rho + \rho.\dot{\varphi}\,\hat{e}_\varphi + \dot{z}\,\hat{e}_z$$

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{\rho}\,\hat{e}_\rho + \rho.\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z\right)$$

$$= \ddot{\rho}\,\hat{e}_\rho + \dot{\rho}.\dot{\phi}\,\hat{e}_\phi + \dot{\rho}\,\dot{\phi}\,\hat{e}_\phi + \rho.\ddot{\phi}\,\hat{e}_\phi + \rho.\dot{\phi}.\dot{\phi}\,\hat{e}_\rho + \ddot{z}\,\hat{e}_z$$

$$= \ddot{\rho}\,\hat{e}_\rho + 2\dot{\rho}\,\dot{\phi}\,\hat{e}_\phi + \rho(\ddot{\phi} + \dot{\phi}\dot{\phi})\hat{e}_\rho + \ddot{z}\,\hat{e}_z$$

⑤ Defined arc length and volume element in cylindrical coordinate system.

Soln.- Arc length $= \sqrt{h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2}$

$$u_1 = \rho\ ,\ u_2 = \phi\ ,\ u_3 = z$$

$$h = 1\ ,\ h_2 = \rho\ ,\ h_3 = 1$$

arc length $= \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2}$

volume element $= h_1 h_2 h_3\, du_1 du_2 du_3$

$$= 1.\rho.1.\,d\rho\, d\phi\, dz$$

$$= \rho\, d\rho\, d\phi\, dz$$

* Gradient in curvilinear coordinate system:—

Let, $\nabla\phi = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$

we know that,

$$dr = \frac{\partial r}{\partial u_1}du_1 + \frac{\partial r}{\partial u_2}du_2 + \frac{\partial r}{\partial u_3}du_3$$

$$= h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3$$

Again, $d\theta = \nabla\varphi \cdot d\vec{r}$

$$= (f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3)\cdot(h_1 du_1\hat{e}_1 + h_2 du_2\hat{e}_2 + h_3 du_3\hat{e}_3)$$

$$= f_1h_1du_1 + f_2h_2du_2 + f_3h_3du_3$$

$$\Rightarrow \frac{\partial\varphi}{\partial u_1}du_1 + \frac{\partial\varphi}{\partial u_2}du_2 + \frac{\partial\varphi}{\partial u_3}du_3 = f_1h_1du_1 + f_2h_2du_2 + f_3h_3du_3$$

Here, $f_1h_1 = \frac{\partial\varphi}{\partial u_1}$, $f_2h_2 = \frac{\partial\varphi}{\partial u_2}$, $f_3h_3 = \frac{\partial\varphi}{\partial u_3}$

$$\Rightarrow f_1 = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}, \Rightarrow f_2 = \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}, \quad f_3 = \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}$$

$$\because \nabla\varphi = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

$$= \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}\hat{e}_3$$

$$= \left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right)\varphi$$

$$\therefore \nabla = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}$$

(6) Express $\nabla\varphi$ in Cylindrical coordinate.

<u>Sol^n.</u> $\nabla\phi = \left(\hat{e}_\rho\frac{\partial}{\partial\rho} + \frac{\hat{e}_\varphi}{\rho}\frac{\partial}{\partial\varphi} + \hat{e}_z\frac{\partial}{\partial z}\right)\phi$

$$= \frac{\partial\phi}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial\phi}{\partial\phi}\hat{e}_\varphi + \frac{\partial\phi}{\partial z}\hat{e}_z$$

Divergane cc :-

$$\bar{\nabla}\cdot\bar{F}$$

$$\bar{F} = F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3$$

$$\bar{\nabla}\cdot\bar{F} = \bar{\nabla}\cdot(F_1\hat{e}_1) + \bar{\nabla}\cdot(F_2\hat{e}_2) + \bar{\nabla}\cdot(F_3\hat{e}_3)$$

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{e}_2}{h_2}\times\frac{\hat{e}_3}{h_3}$$

$$\nabla u_1 = \frac{\hat{e}_1}{h_1}$$

$$\nabla u_2 = \frac{\hat{e}_2}{h_2}$$

$$\bar{\nabla} u_3 = \frac{\hat{e}_3}{h_3}$$

$$\bar{\nabla} u_2 \times \bar{\nabla} u_3 = \frac{\hat{e}_1}{h_2 h_3}$$

$$\hat{e}_1 = h_2 h_3(\nabla u_2 \times \nabla u_3)$$

$$\nabla\cdot(F_1\hat{e}_1) = \bar{\nabla}\cdot\left[\overbrace{F_1 h_2 h_3}^{\varphi}\overbrace{(\nabla u_2 \times \nabla u_3)}^{\psi}\right]$$

$$\nabla\cdot(\varphi\bar{\psi}) = \bar{\nabla}\varphi\cdot\bar{\psi} + \varphi\,\bar{\nabla}\cdot\psi$$

$$= \vec{\nabla}(F_1 h_2 h_3)\cdot(\bar{\nabla}u_2 \times \bar{\nabla}u_3) + F_1 h_2 h_3\,\underbrace{\vec{\nabla}\cdot(\bar{\nabla}u_2 \times \bar{\nabla}u_3)}_{0}$$

$$\nabla(F_1\hat{e}_1) = \bar{\nabla}(F_1 h_2 h_3)\cdot(\nabla u_2 \times \bar{\nabla}u_2)$$

$$= \bar{\nabla}(f_1 h_2 h_3)\cdot\frac{\hat{e}_1}{h_2 h_3}$$

$$\bar{\nabla}(F_1\hat{e}_1) = \frac{\hat{e}_1}{h_2 h_3}\cdot\bar{\nabla}(f_1 h_2 h_3)$$

$$\bar{\nabla}(F_2\hat{e}_2) = \frac{\hat{e}_2}{h_3 h_1}\cdot\bar{\nabla}(f_2 h_3 h_1)$$

$$\nabla(F_3\hat{e}_3) = \frac{\hat{e}_3}{h_1 h_2}\cdot\bar{\nabla}(f_3 h_1 h_2)$$

$$\vec{\nabla}\cdot(F_1\hat{e}_1) = \frac{\hat{e}_1}{h_2 h_3}\cdot\left[\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\cdot\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\cdot\frac{\partial}{\partial u_3}\right](f_1 h_2 h_3)$$

$$= \frac{1}{h_2 h_3 h_1}\frac{\partial}{\partial u_1}(f_1 h_2 h_3)$$

$\bar{\nabla}(f_2\hat{e}_2) =$

$$\bar{\nabla}\cdot\bar{F} = \bar{\nabla}(f_1\hat{e}_1) + \bar{\nabla}(f_2\hat{e}_2) + \nabla(f_3\hat{e}_3)$$

$$= \frac{1}{h_2h_3h_1}\frac{\partial}{\partial u}(f_1h_2h_3) + \frac{1}{h_2h_3h_1}\frac{\partial}{\partial u_2}(f_2h_3h_1) + \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}(f_2h_1h_2)$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(f_1h_2h_3) + \frac{\partial}{\partial u_2}(f_2h_3h_1) + \frac{\partial}{\partial u_3}(f_3h_1h_3)\right]$$

~~Qust~~ :- Laplacian operator (∇^2) :-

$$\bar{\nabla}\psi = \frac{\hat{e}_1}{h_1}\frac{\partial\psi}{\partial u_1} + \frac{\hat{e}_2}{h_3}\cdot\frac{\partial\psi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\psi}{\partial u_3} \quad \text{——(1)}$$

$$\bar{\nabla}\psi = \bar{F} = F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3 \quad \text{——(2)}$$

From (1) & (2)

$$F_1 = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1},\; F_2 = \frac{1}{h_2}\frac{\partial\psi}{\partial u_2},\; F_3 = \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}$$

$$\nabla^2\psi = \bar{\nabla}\cdot\bar{\nabla}\psi = \bar{\nabla}\cdot\bar{F}$$

$$\bar{\nabla}\cdot\bar{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(f_1h_2h_3) + \frac{\partial}{\partial u_2}(f_2h_1h_3) + \frac{\partial}{\partial u_3}(f_3h_1h_2)\right]$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\right)h_2h_3 + \frac{\partial}{\partial u_2}\left(\frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\right)h_1h_3 + \frac{\partial}{\partial u_3}\left(\frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\right)h_1h_2\right]$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{\partial u_3}\frac{\partial \psi}{\partial u_3}\right)\right]\psi$$

$$= \nabla^2 \psi$$

Curl :—

$$\bar{F} = \bar{F}(u_1, u_2, u_3)$$

$$\bar{F} = F_1 \hat{e}_1 + F_2 \hat{e}_2 + F_3 \hat{e}_3 \quad \text{——— (1)}$$

$$\text{Curl } \bar{F} = \bar{\nabla} \times \bar{F} = \bar{\nabla} \times (F_1 \hat{e}_1 + F_2 \hat{e}_2 + F_3 \hat{e}_3)$$

$$\bar{\nabla} \times \vec{F} = \bar{\nabla} \times (F_1 \hat{e}_1) + \bar{\nabla} \times (F_2 \hat{e}_2) + \vec{\nabla}(F_3 \hat{e}_3)$$

$$\bar{\nabla} u_1 = \frac{\hat{e}_1}{h_1}$$

$$\hat{e}_1 = h_1 \bar{\nabla} u_1$$

$$\bar{\nabla} \times (F_1 \hat{e}_1) = \bar{\nabla} \times (\underbrace{F_1 h_1}_{\phi} \underbrace{\bar{\nabla} u_1}_{\bar{A}})$$

$$\bar{\nabla} \times (\phi \vec{A}) = \phi(\bar{\nabla} \times A) + (\nabla \phi) \times A$$

$$\bar{\nabla} \times (F_1 \hat{e}_1) = F_1 h_1 \underbrace{(\bar{\nabla} \times \bar{\nabla} u_1)}_{0} + (\bar{\nabla} F h_1) \times \bar{\nabla} u_1$$

$$= \bar{\nabla}(F_1 h_1) \times \bar{\nabla} u_1$$

$$= \bar{\nabla}(F_1 h_1) \times \frac{\hat{e}_1}{h_1}$$

$$\cancel{\bar{\nabla} \times (F_1 h_1)}\ \bar{\nabla} \times (F_1 \hat{e}_1) = \left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right)(F_1 h_1) \times \frac{\hat{e}_1}{h_1}$$

$$= \left(\frac{\hat{e}_1}{h_1}\frac{\partial (F_1 h_1)}{\partial u_1} \times \frac{\hat{e}_1}{h_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2}(F_1 h_1) \times \frac{\hat{e}_1}{h_1} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}(F_1 h_1) \times \frac{\hat{e}_1}{h_1}\right)$$

$$\bar{\nabla} \times (F_1 \hat{e}_1) = -\frac{\hat{e}_3}{h_2 h_1} \frac{\partial (F_1 h_1)}{\partial u_2} + \frac{\hat{e}_2}{h_3 h_1} \frac{\partial}{\partial u_3} (F_1 h_1)$$

$$= \frac{\hat{e}_2}{h_3 h_1} \frac{\partial}{\partial u_3} (F_1 h_1) - \frac{\hat{e}_3}{h_2 h_1} \frac{\partial (F_1 h_1)}{\partial u_2}$$

$$\bar{\nabla} \times F_2 \hat{e}_2 = \frac{\hat{e}_3}{h_1 h_2} \frac{\partial}{\partial u_1} (F_2 h_2) - \frac{\hat{e}_1}{h_2 h_3} \frac{\partial (F_2 h_2)}{\partial u_2}$$

$$\bar{\nabla} \times F_3 \hat{e}_3 = \frac{\hat{e}_1}{h_2 h_3} \frac{\partial (F_3 h_3)}{\partial u_3} - \frac{\hat{e}_2}{h_3 h_1} \frac{\partial (F_3 h_3)}{\partial u_1}$$

$$\vec{\nabla} \times \vec{F} = \cancel{\hat{e}_1} \frac{\hat{e}_1}{h_2 h_3} \left(\frac{\partial (F_3 h_3)}{\partial u_2} - \frac{\partial (F_2 h_2)}{\partial u_3} \right) + \frac{\hat{e}_2}{h_3 h_1} \left(\frac{\partial (F_1 h_1)}{\partial u_1} - \frac{\partial}{\partial u_1} (F_3 h_3) \right) + \frac{\hat{e}_3}{h_1 h_2} \left(\frac{\partial}{\partial u_1} (F_2 h_2) - \frac{\partial}{\partial u_2} (F_1 h_1) \right)$$

$$= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}$$