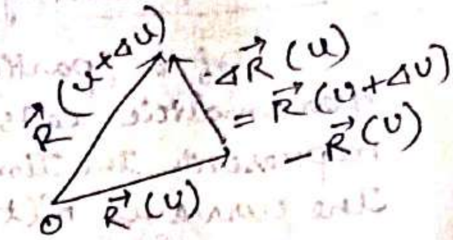
Vector Analysis

Vector Differentiation :-

Ordinary derivatives of vector-valued funcⁿ :-

Let $\vec{R}(u)$ be a vector depending on a single variable u. Then $\frac{\Delta \vec{R}}{\Delta u} = \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}$

where Δu denotes an increment in u.



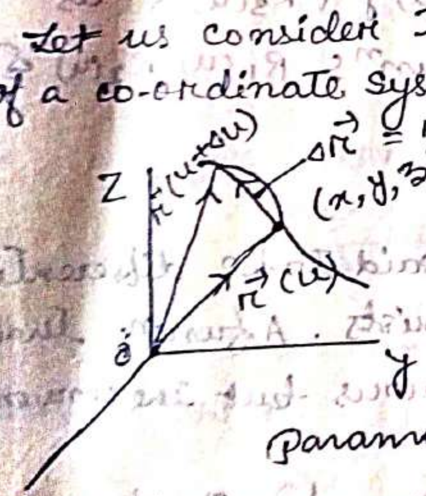
The ordinary derivative of the vector $\vec{R}(u)$ with respect to the scalar u is given as follows when the limit exists.

$$\frac{d\vec{R}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \vec{R}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}$$

Since $\frac{d\vec{R}}{du}$ is itself a vector depending on u, its derivative w.r.t u can be considered a vector too. If this derivative exists, it is denoted by $\frac{d^2R}{du^2}$ and so on for higher order derivatives.

Space Curves

Let us consider the position vector $\vec{r}(u)$ joining the origin O of a co-ordinate system and any point (x, y, z)



Then $\vec{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$ — ①

As u changes to $u+\Delta u$, the terminal point of $\vec{r}$ describes a space curve having the paramatic eqⁿs.

$$x = x(u)$$
$$y = y(u)$$
$$z = z(u)$$

Then $\frac{\Delta \vec{r}}{\Delta u} = \frac{\vec{r}(u+\Delta u) - \vec{r}(u)}{\Delta u}$ is a vector in the direcⁿ of $\Delta \vec{r}$ or opposite to it according as $\Delta u > 0$ or $\Delta u < 0$

If the limit exists,

$$\lim_{\Delta u \to 0} \frac{\Delta \vec{r}}{\Delta u} = \frac{d\vec{r}}{du}$$

where $\frac{d\vec{r}}{du}$ is a vector in a direcn tangent to the space curve at (x, y, z) such that

$$\frac{d\vec{r}}{du} = \frac{dx}{du}\hat{i} + \frac{dy}{du}\hat{j} + \frac{dz}{du}\hat{k}$$

- Motion, velocity and acceleration :-

If a particle P moves along a space curve C whose paramattric eqns are $x = x(t)$, $y = y(t)$ and $z = z(t)$ where t represents the time, then the position vector of particle P along the curve is $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

The velocity $\vec{v}$ and accn $\vec{a}$ of the particle P is given by

$$\vec{v} = \vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{a} = \vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

Continuity and differentiabity

A scalar funcn $\phi(u)$ is said to be continuous at u if $\lim_{\Delta u \to 0} \phi(u + \Delta u) = \phi(u)$

A vector funcn $\vec{R}(u) = R_1(u)\hat{i} + R_2(u)\hat{j} + R_3(u)\hat{k}$ is said to be continuous at u if the 3 funns $R_1(u)$, $R_2(u)$ & $R_3(u)$ are continuous at u or if

$$\lim_{\Delta u \to 0} \vec{R}(u + \Delta u) = \vec{R}(u)$$

A scalar or vector funcn of u is said to be differentiable upto order n if its nth derivative exists. A funcn that is differentiable is necessarily continuous but the converse is not true.

If $\vec{A}$, $\vec{B}$ & $\vec{C}$ are differentiable vector funcns of a scalar u and if ϕ is a differentiable scalar funcn of u, then

(1) $\frac{d}{du}(\vec{A} + \vec{B}) = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$

(2) $\frac{d}{du}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{d\vec{B}}{du} + \frac{d\vec{A}}{du} \cdot \vec{B}$

(3) $\frac{d}{du}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{du} + \frac{d\vec{A}}{du}\times\vec{B}$

(4) $\frac{d}{du}(\phi\vec{A}) = \phi\frac{d\vec{A}}{du} + \frac{d\phi}{du}\vec{A}$

(5) $\frac{d}{du}(\vec{A}\cdot\vec{B}\times\vec{C}) = \frac{d\vec{A}}{du}\cdot\vec{B}\times\vec{C} + \vec{A}\cdot\frac{d\vec{B}}{du}\times\vec{C} + \vec{A}\cdot\vec{B}\times\frac{d\vec{C}}{du}$

(6) $\frac{d}{du}[\vec{A}\times(\vec{B}\times\vec{C})] = \frac{d\vec{A}}{du}\times(\vec{B}\times\vec{C}) + \vec{A}\times(\frac{d\vec{B}}{du}\times\vec{C}) + \vec{A}\times(\vec{B}\times\frac{d\vec{C}}{du})$

Partial derivation of vectors

If A is a vector depending on more than one variable, i.e let $\vec{A}=\vec{A}(x,y,z)$ then the partial derivative of $\vec{A}$ w.r.t x is as follows.

when the limit exists,

$$\frac{\delta\vec{A}}{\delta x} = \lim_{\Delta x\to 0}\frac{\vec{A}(x+\Delta x,y,z)-\vec{A}(x,y,z)}{\Delta x}$$

Similarly,
$$\frac{\delta\vec{A}}{\delta y} = \lim_{\Delta y\to 0}\frac{\vec{A}(x,y+\Delta y,z)-\vec{A}(x,y,z)}{\Delta y}$$

$$\frac{\delta\vec{A}}{\delta z} = \lim_{\Delta z\to 0}\frac{\vec{A}(x,y,z+\Delta z)-\vec{A}(x,y,z)}{\Delta z}$$

$\phi(x,y)$ is said to be continuous at (x,y) if

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\phi(x+\Delta x, y+\Delta y) = \phi(x,y)$$

Higher derivatives are as follows;

$$\frac{\delta^2\vec{A}}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta\vec{A}}{\delta x}\right),\quad \frac{\delta^2 A}{\delta x\delta y} = \frac{\delta}{\delta x}\left(\frac{\delta\vec{A}}{\delta y}\right) \text{ and so on}$$

If $\vec{A}$ has continuous partial derivatives of second order (say), then

$$\frac{\delta^2\vec{A}}{\delta x\delta y} = \frac{\delta^2\vec{A}}{\delta y\delta x}$$ i.e. the order of differentiation does not matter.

Let $\vec{A}$ & $\vec{B}$ vector functions of (x,y,z). Then

$$\frac{\delta}{\delta x}(\vec{A}\cdot\vec{B}) = \frac{\delta\vec{A}}{\delta x}\cdot\vec{B} + \vec{A}\cdot\frac{\delta\vec{B}}{\delta x}$$

② $\frac{\partial}{\partial x}(\vec{A}\times\vec{B}) = \frac{\partial \vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial \vec{B}}{\partial x}$

③ $\frac{\partial^2}{\partial y\,\partial x}(\vec{A}\cdot\vec{B}) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})\right]$

$= \frac{\partial}{\partial y}\left[\frac{\partial \vec{A}}{\partial x}\cdot\vec{B} + \vec{A}\cdot\frac{\partial \vec{B}}{\partial x}\right]$

$= \frac{\partial^2 \vec{A}}{\partial y\,\partial x}\cdot\vec{B} + \frac{\partial \vec{A}}{\partial x}\cdot\frac{\partial \vec{B}}{\partial y} + \frac{\partial \vec{A}}{\partial y}\cdot\frac{\partial \vec{B}}{\partial x} + \vec{A}\cdot\frac{\partial^2 \vec{B}}{\partial y\,\partial x}$

The rules for the differentiation of vector are the same as in elementary calculus. They are as follows. If $\vec{A}$ & $\vec{B}$ are vector funcns of x, y, z, then

1) $\vec{A} = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$

$\Rightarrow d\vec{A} = \hat{i}\,dA_1 + \hat{j}\,dA_2 + \hat{k}\,dA_3$

2) $d(\vec{A}\cdot\vec{B}) = d\vec{A}\cdot\vec{B} + \vec{A}\cdot d\vec{B}$

3) $d(\vec{A}\times\vec{B}) = d\vec{A}\times\vec{B} + \vec{A}\times d\vec{B}$

If $\vec{A} = \vec{A}(x,y,z)$ Then,

$d\vec{A} = \frac{\partial \vec{A}}{\partial x}dx + \frac{\partial \vec{A}}{\partial y}dy + \frac{\partial \vec{A}}{\partial z}dz$

Problem 1

Let $\vec{R}(u) = \hat{i}x(u) + \hat{j}y(u) + \hat{k}z(u)$ where x, y, z are differentiable functions of a scalar variable u.

Prove that $\frac{d\vec{R}}{du} = \hat{i}\frac{dx}{du} + \hat{j}\frac{dy}{du} + \hat{k}\frac{dz}{du}$

Soln:- $\vec{R}(u+\Delta u) = (x+\Delta x)\hat{i} + (y+\Delta y)\hat{j} + (z+\Delta z)\hat{k}$

$\therefore \frac{\Delta \vec{R}}{\Delta u} = \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}$

$= \frac{(x+\Delta x)\hat{i} + (y+\Delta y)\hat{j} + (z+\Delta z)\hat{k} - x\hat{i} - y\hat{j} - z\hat{k}}{\Delta u}$

$\therefore \frac{\Delta \vec{R}}{\Delta u} = \left\{\frac{(x+\Delta x) - x}{\Delta u}\right\}\hat{i} + \left\{\frac{(y+\Delta y) - y}{\Delta u}\right\}\hat{j} + \left\{\frac{(z+\Delta z) - z}{\Delta u}\right\}\hat{k}$

$$\frac{d\vec{R}}{du} = \lim_{\Delta u \to 0}\left\{\frac{\Delta x}{\Delta u}\hat{i} + \frac{\Delta y}{\Delta u}\hat{j} + \frac{\Delta z}{\Delta u}\hat{k}\right\}$$

$$\therefore \frac{d\vec{R}}{du} = \frac{dx}{du}\hat{i} + \frac{dy}{du}\hat{j} + \frac{dz}{du}\hat{k}$$

2. A particle moves along a curve $x = 2t^2$, $y = t^2 - 4t$, $z =$ where t is the time. Find the components of its veloci and acceleration at time $t=1$ in the direc^n $\hat{i} - 2\hat{j} + 2\hat{k}$

Soln:

$$v = \frac{d\vec{r}}{dt} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}$$

$$= \hat{i}\frac{d}{dt}(2t^2) + \hat{j}\frac{d}{dt}(t^2 - 4t) + \hat{k}\frac{d}{dt}(-t-5)$$

$$= \hat{i}\,4t + \hat{j}\,(2t-4) + \hat{k}\,(-1)$$

$$= 4t\hat{i} + (2t-4)\hat{j} - \hat{k}$$

$$t=1 \Rightarrow v = 4\hat{i} - 2\hat{j} - \hat{k}$$

unit vector along the direc^n $\hat{i} - 2\hat{j} + 2\hat{k}$ $= \dfrac{\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{1^2 + (2)^2 + 2^2}}$

$$= \frac{\hat{i} - 2\hat{j} + \hat{k}}{3}$$

$$= \frac{\hat{i}}{3} - \frac{2}{3}\hat{j} + \frac{\hat{k}}{3}$$

component of $\vec{v}$ along $\hat{i} - 2\hat{j} + 2\hat{k}$

$$(4\hat{i} - 2\hat{j} - \hat{k})\cdot\left(\frac{\hat{i}}{3} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$

$$= \frac{4}{3} + \frac{4}{3} - \frac{2}{3}$$

$$= \frac{6}{3}$$

$$= 2$$

$$a = \frac{d^2\vec{r}}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}\left\{4t\hat{i} + (2t-4)\hat{j} - \hat{k}\right\}$$

$$= (\hat{i}\cdot 4 + 2\hat{j}$$

$$= 4\hat{i} + 2\hat{j}$$

Component of $\vec{a}$ along $\hat{i} - 2\hat{j} + 2\hat{k}$

$$(4\hat{i} + 2\hat{j})\cdot\left(\frac{\hat{i}}{3} - \frac{2}{3}\hat{j} + \frac{\hat{k}}{3}\right)$$

$$= \frac{4}{3} - \frac{4}{3}$$

$$= 0$$

Q) A curve c is defined by the parametric eqⁿs. $x = x(s)$, $y = y(s)$, $z = z(s)$ where s is the arc length of c measured from a fixed point on c.

If $\vec{r}$ is the position vector of any point on c, show that $\frac{d\vec{r}}{ds}$ is a unit vector tangent to c.

Soln:

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$\therefore \frac{d\vec{r}}{ds} = \frac{d}{ds}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \hat{i}\frac{dx}{ds} + \hat{j}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}$$

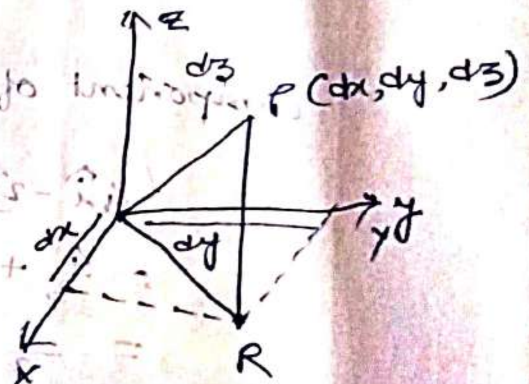
is a tangent to the curve $x = x(s)$, $y = y(s)$, $z = z(s)$

$$\therefore \left|\frac{d\vec{r}}{ds}\right| = \cancel{\left|\hat{i}\frac{dx}{ds} + \hat{j}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}\right|}$$

$$= \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$$

but $(dx)^2 + (dy)^2 + (dz)^2 = ds^2$

$$\therefore \left|\frac{d\vec{r}}{ds}\right| = 1$$



Proof :-

① $$\frac{d}{du}(\vec{A} + \vec{B}) = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$$

$$\frac{d}{du}(\vec{A} + \vec{B}) = \lim_{\Delta u \to 0} \frac{\vec{A}(u+\Delta u) + \vec{B}(u+\Delta u) - \vec{A}(u) - \vec{B}(u)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\vec{A}(u+\Delta u) - \vec{A}(u)}{\Delta u} + \lim_{\Delta u \to 0} \frac{\vec{B}(u+\Delta u) - \vec{B}(u)}{\Delta u}$$

$$= \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$$

//

② $\frac{d}{du}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{d\vec{B}}{du} + \frac{d\vec{A}}{du}\cdot\vec{B}$

L.H.S $= \frac{d}{du}(\vec{A}\cdot\vec{B})$

$= \frac{d}{du}(A_1B_1 + A_2B_2 + A_3B_3)$ if $\vec{A} = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$, $\vec{B} = \hat{i}B_1 + \hat{j}B_2 + \hat{k}B_3$

Now, $\frac{d}{du}(A_1B_1) = \lim_{\Delta u \to 0} \frac{A_1(u+\Delta u)\cdot B_1(u+\Delta u) - A_1(u)B_1(u)}{\Delta u}$

$= \lim_{\Delta u \to 0} \frac{A_1(u+\Delta u)B_1(u+\Delta u) - A_1(u)B_1(u+\Delta u) + A_1(u)B_1(u+\Delta u) - A_1(u)B_1(u)}{\Delta u}$

$= \lim_{\Delta u \to 0}\left\{\frac{\{A_1(u+\Delta u) - A_1(u)\}B_1(u+\Delta u)}{\Delta u}\right\} + \lim_{\Delta u \to 0}\left\{\frac{A_1(u)\{B(u+\Delta u) - B_1(u)\}}{\Delta u}\right\}$

$= \frac{d}{du}A_1 B_1(u) + A_1(u)\frac{d}{du}B_1$

$\frac{d}{du}(A_1B_1) = \frac{d}{du}A_1 B_1 + A_1\frac{dB_1}{du}$

$\frac{d}{du}(A_2B_2) = \frac{dA_2}{du}B_2 + A_2\frac{dB_2}{du}$

$\frac{d}{du}(A_3B_3) = \frac{dA_3}{du}B_3 + A_3\frac{dB_3}{du}$

L.H.S $= \frac{dA_1}{du}B_1 + \frac{d}{du}A_2 B_2 + \frac{dA_3}{du}B_3 + A_1\frac{dB_1}{du} + A_2\frac{dB_2}{du} + A_3\frac{dB_3}{du}$

$= \left(\hat{i}\frac{dA_1}{du} + \hat{j}\frac{dA_2}{du} + \hat{k}\frac{dA_3}{du}\right)\cdot(\hat{i}B_1 + \hat{j}B_2 + \hat{k}B_3) + (\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3)\cdot\left(\hat{i}\frac{dB_1}{du} + \hat{j}\frac{dB_2}{du} + \hat{k}\frac{dB_3}{du}\right)$

$= \frac{d}{du}(\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3)\cdot(\hat{i}B_1 + \hat{j}B_2 + \hat{k}B_3) + (\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3)\cdot\frac{d}{du}(\hat{i}B_1 + \hat{j}B_2 + \hat{k}B_3)$

$= \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}$

$=$ R.H.S //

3) $\frac{d}{du}(\phi\vec{A}) = \frac{d\phi}{du}\vec{A} + \phi\frac{d\vec{A}}{du}$

L.H.S $= \frac{d}{du}(\phi\vec{A})$

$= \frac{d}{du}\phi(\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3)$

$= \hat{i}\frac{d}{du}(\phi A_1) + \hat{j}\frac{d}{du}(\phi A_2) + \hat{k}\frac{d}{du}(\phi A_3)$

$= \hat{i}(\phi\frac{dA_1}{du} + \frac{d\phi}{du}A_1) + \hat{j}(\phi\frac{dA_2}{du} + \frac{d\phi}{du}A_2) + \hat{k}(\phi\frac{dA_3}{du} + \frac{d\phi}{du}A_3)$

$= \phi\frac{d}{du}(\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3) + \frac{d\phi}{du}(\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3)$

$= \phi\frac{d\vec{A}}{du} + \frac{d\phi}{du}\vec{A}$

= R.H.S //

4) $\frac{d}{du}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{du}\times\vec{B} + \frac{d\vec{B}}{du}\times\vec{A}$

L.H.S $= \frac{d}{du}(\vec{A}\times\vec{B})$

$= \lim_{\Delta u\to 0}\frac{(A+\Delta A)\times(B+\Delta B) - A\times B}{\Delta u}$

$= \lim_{\Delta u\to 0}\frac{\cancel{A\times B} + A\times\Delta B + \Delta A\times B + \Delta A\times\Delta B - \cancel{A\times B}}{\Delta u}$

$= \lim_{\Delta u\to 0}\frac{A\times\Delta B + \Delta A\times B + \Delta A\times\Delta B}{\Delta u}$

$= \lim_{\Delta u\to 0} A\times\frac{\Delta B}{\Delta u} + \frac{\Delta A}{\Delta u}\times B + \frac{\Delta A\times\Delta B}{\Delta u}$

$= A\times\frac{dB}{du} + \frac{dA}{du}\times B$

= R.H.S //

(5) $\frac{d}{du}(A\cdot B\times C) = A\cdot B\times\frac{dC}{du} + A\cdot\frac{dB}{du}\times C + \frac{dA}{du}\cdot B\times C$

$$\text{L.H.S} = \frac{d}{du}(A\cdot B\times C)$$

$$= A\cdot\frac{d}{du}(B\times C) + \frac{dA}{du}\cdot B\times C$$

$$= A\cdot\left[B\times\frac{dC}{du} + \frac{dB}{du}\times C\right] + \frac{dA}{du}\cdot B\times C$$

$$= A\cdot B\times\frac{dC}{du} + A\cdot\frac{dB}{du}\times C + \frac{dA}{du}\cdot B\times C$$

$$= \text{R.H.S} \quad //$$

Q) if $\vec{A}$ has constant magnitude show that $\vec{A}\cdot\frac{d\vec{A}}{dt} = 0$ and that $\vec{A}$ and $\frac{d\vec{A}}{dt}$ are perpendicular to each other. provided $\left|\frac{d\vec{A}}{dt}\right| \neq 0$

Soln:

$$\frac{d}{dt}(\vec{A}\cdot\vec{A}) = \vec{A}\cdot\frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{A}$$

$$= 2\vec{A}\cdot\frac{d\vec{A}}{dt} \quad \text{--- (1)}$$

Again $\frac{d}{dt}(\vec{A}\cdot\vec{A}) = \frac{d}{dt}A^2$

$$= \cancel{2A}\ 2A\frac{dA}{dt}$$

$$= 0 \quad \text{--- (2)}$$

Since A is const.

from (1) & (2) $\vec{A}\cdot\frac{d\vec{A}}{dt} = 0$

Now, $\vec{A}\cdot\frac{d\vec{A}}{dt} = 0 \Rightarrow \vec{A}$ & $\vec{B}$ are perpendicular if $|\vec{A}| \neq 0$, $|\vec{B}| \neq 0$

$\therefore \vec{A}\cdot\frac{d\vec{A}}{dt} = 0$

$\Rightarrow \vec{A}$ & $\frac{d\vec{A}}{dt}$ are perpendicular provided $\left|\frac{d\vec{A}}{dt}\right| \neq 0$

Q) Show that $\vec{A}\cdot\frac{d\vec{A}}{dt} = A\frac{dA}{dt}$

Soln:

$$\frac{d}{dt}(\vec{A}\cdot\vec{A}) = 2\vec{A}\cdot\frac{d\vec{A}}{dt} \longrightarrow (1)$$

also $\frac{d}{dt}(\vec{A}\cdot\vec{A}) = \frac{d}{dt}A^2$

$$= 2A\frac{dA}{dt} \rightarrow (2)$$

from (1) & (2)

$$\vec{A}\cdot\frac{d\vec{A}}{dt} = A\frac{dA}{dt}$$ //

Alter. method

$$A = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

$$\frac{dA}{dt} = \frac{1}{2\sqrt{A_1^2+A_2^2+A_3^2}} \cdot 2A_1 \frac{dA_1}{dt} + 2A_2 \frac{dA_2}{dt} + 2A_3 \frac{dA_3}{dt}$$

$$= A^{-1}\left(A_1 \frac{dA_1}{dt} + A_2 \frac{dA_2}{dt} + A_3 \frac{dA_3}{dt}\right)$$

$$\Rightarrow A\frac{dA}{dt} = (\hat{i} A_1 + \hat{j} A_2 + \hat{k} A_3)\cdot\left(\hat{i}\frac{dA_1}{dt} + \hat{j}\frac{dA_2}{dt} + \hat{k}\frac{dA_3}{dt}\right)$$

$$= \vec{A} \cdot \frac{d\vec{A}}{dt}$$

3) Let $\vec{F}$ depend on x, y, z, t where x, y & z depend on t. Prove That

$$\frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + \frac{\partial \vec{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{F}}{\partial z}\frac{dz}{dt}$$

Soln: Let $\vec{F}(x,y,z,t) = \hat{i} F_1(x,y,z,t) + \hat{j} F_2(x,y,z,t) + \hat{k} F_3(x,y,z,t)$

$$\therefore \quad d\vec{F} = \hat{i}\, dF_1 + \hat{j}\, dF_2 + \hat{k}\, dF_3$$

$$d\vec{F} = \hat{i}\left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz + \frac{\partial F_1}{\partial t}dt\right) +$$

$$\hat{j}\left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz + \frac{\partial F_2}{\partial t}dt\right)$$

$$+\hat{k}\left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz + \frac{\partial F_3}{\partial t}dt\right)$$

$$d\vec{F} = \left(\hat{i}\frac{\partial F_1}{\partial x} + \hat{j}\frac{\partial F_2}{\partial x} + \hat{k}\frac{\partial F_3}{\partial x}\right)dx + \left(\hat{i}\frac{\partial F_1}{\partial y} + \hat{j}\frac{\partial F_2}{\partial y} + \hat{k}\frac{\partial F_3}{\partial y}\right)dy + \left(\hat{i}\frac{\partial F_1}{\partial z} + \hat{j}\frac{\partial F_2}{\partial z} + \hat{k}\frac{\partial F_3}{\partial z}\right)dz + \left(\hat{i}\frac{\partial F_1}{\partial t} + \hat{j}\frac{\partial F_2}{\partial t} + \hat{k}\frac{\partial F_3}{\partial t}\right)dt$$

$$= \frac{\partial \vec{F}}{\partial x}dx + \frac{\partial \vec{F}}{\partial y}dy + \frac{\partial \vec{F}}{\partial z}dz + \frac{\partial \vec{F}}{\partial t}\cdot dt$$

$$\frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + \frac{\partial \vec{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{F}}{\partial z}\frac{dz}{dt}$$

Gradient, Divergence & curl :-

$\vec{\nabla}$ Del operator

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$ ($\vec{\nabla}$) is called 'nabla'

$\vec{\nabla}\phi(x,y,z) = \text{Grad } \phi$
$= \text{Gradient of } \phi$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\vec{A}(x,y,z)$

$\vec{\nabla}\cdot\vec{A}$ = Divergence of $\vec{A}$

$\vec{\nabla}\times\vec{A}$ = curl of $\vec{A}$

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z\right)$$
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{curl } \vec{A} = \vec{\nabla}\times\vec{A}$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z\right)$$
$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Formulae involving $\vec{\nabla}$

Let $\vec{A}$ & $\vec{B}$ be differentiable vector functions and let ϕ & ψ be differentiable scalar functions of posn (x,y,z).

Then ① $\vec{\nabla}(\phi+\psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$

2) $\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$

3) $\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$

4) $\vec{\nabla}\cdot\phi\vec{A} = \vec{\nabla}\phi\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$

5) $\vec{\nabla}\times\phi\vec{A} = \vec{\nabla}\phi\times\vec{A} + \phi(\vec{\nabla}\times\vec{A})$

6) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

7) $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B})$

8) $\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{B}\times(\vec{\nabla}\times\vec{A}) + \vec{A}\times(\vec{\nabla}\times\vec{B})$

if ϕ & $\vec{A}$ have continuous second partial derivatives, then,

9) $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$

10) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

13) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

12) $\vec{\nabla}\cdot\vec{\nabla}\phi = \nabla^2\phi$

$$= \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

is called the Laplacian operator.

① Prove $\vec{\nabla}(\phi+\psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$

$$\vec{\nabla}(\phi+\psi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi+\psi)$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi+\psi) + \hat{j}\frac{\partial}{\partial y}(\phi+\psi) + \hat{k}\frac{\partial}{\partial z}(\phi+\psi)$$

$$= \hat{i}\frac{\partial}{\partial x}\phi + \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} + \hat{k}\frac{\partial\psi}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x}\phi + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) + \left(\frac{\partial}{\partial x}\psi\,\hat{i} + \frac{\partial}{\partial y}\psi\,\hat{j} + \frac{\partial}{\partial z}\psi\,\hat{k}\right)$$

$$= \vec{\nabla}\phi + \vec{\nabla}\psi$$

(2) $\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$

$$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{\hat{i}(A_x+B_x)+\hat{j}(A_y+B_y)+\hat{k}(A_z+B_z)\right\}$$

$$= \frac{\partial}{\partial x}(A_x+B_x)+\frac{\partial}{\partial y}(A_y+B_y)+\frac{\partial}{\partial z}(A_z+B_z)$$

$$= \frac{\partial}{\partial x}A_x+\frac{\partial}{\partial x}B_x+\frac{\partial}{\partial y}A_y+\frac{\partial}{\partial y}B_y+\frac{\partial}{\partial z}A_z+\frac{\partial}{\partial z}B_z$$

$$= \left(\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial y}A_y+\frac{\partial}{\partial z}A_z\right)+\left(\frac{\partial}{\partial x}B_x+\frac{\partial}{\partial y}B_y+\frac{\partial}{\partial z}B_z\right)$$

$$= \vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$$

(3) $\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$

$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left\{\hat{i}(A_x+B_x)+\hat{j}(A_y+B_y)+\hat{k}(A_z+B_z)\right\}$$

$$\begin{vmatrix} i & j & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x+B_x & A_y+B_y & A_z+B_z \end{vmatrix}$$

$$= \cancel{i\left\{\frac{\partial}{\partial y}(A_z+B_z)-\frac{\partial}{\partial z}(A_y+B_y)\right\}} + j$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$$

(4) $\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A}+\phi\vec{\nabla}\cdot\vec{A}$

$$\vec{\nabla}\cdot(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{\phi(\hat{i}A_x+\hat{j}A_y+\hat{k}A_z)\right\}$$

$$= \frac{\partial}{\partial x}(\phi A_x)+\frac{\partial}{\partial y}(\phi A_y)+\frac{\partial}{\partial z}(\phi A_z)$$

$$= \phi\frac{\partial A_x}{\partial x}+A_x\frac{\partial\phi}{\partial x}+A_y\frac{\partial\phi}{\partial y}+\phi\frac{\partial A_y}{\partial y}+\phi\frac{\partial A_z}{\partial z}+A_z\frac{\partial\phi}{\partial z}$$

$$= \phi\left(\frac{\partial}{\partial x}A_x + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + A_x\frac{\partial \phi}{\partial x} + A_y\frac{\partial \phi}{\partial y} + A_z\frac{\partial \phi}{\partial z}$$

$$= \phi\, \vec{\nabla}\cdot\vec{A} + (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)\cdot\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= (\phi\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot(\vec{\nabla}\phi))$$

$$= (\vec{\nabla}\phi)\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$$

5) $\vec{\nabla}\times(\phi\vec{A}) = \phi(\vec{\nabla}\times\vec{A}) + (\vec{\nabla}\phi)\times\vec{A}$

$$\vec{\nabla}\times(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\phi(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_x & \phi A_y & \phi A_z \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(\phi A_z) - \frac{\partial}{\partial z}(\phi A_y)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}(\phi A_x) - \frac{\partial}{\partial x}(\phi A_z)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(\phi A_y) - \frac{\partial}{\partial y}(\phi A_x)\right\}$$

$$= \hat{i}\left\{\phi\frac{\partial A_z}{\partial y} + A_z\frac{\partial \phi}{\partial y} - A_y\frac{\partial \phi}{\partial z} - \phi\frac{\partial A_y}{\partial z}\right\}$$
$$+ \hat{j}\left\{\phi\frac{\partial A_x}{\partial z} + A_x\frac{\partial \phi}{\partial z} - \phi\frac{\partial A_z}{\partial x} - A_z\frac{\partial \phi}{\partial x}\right\}$$
$$+ \hat{k}\left\{\phi\frac{\partial A_y}{\partial x} + A_y\frac{\partial \phi}{\partial x} - \phi\frac{\partial A_x}{\partial y} - A_x\frac{\partial \phi}{\partial y}\right\}$$

$$= \phi\left\{\hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right.$$
$$\left. + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right\} + \hat{i}\left(\frac{\partial \phi}{\partial y}A_z - A_y\frac{\partial \phi}{\partial z}\right)$$
$$+ \hat{j}\left(\frac{\partial \phi}{\partial z}A_x - A_z\frac{\partial \phi}{\partial x}\right) + \hat{k}\left(A_y\frac{\partial \phi}{\partial x} - A_x\frac{\partial}{\partial}\right.$$

$$= \phi(\vec{\nabla}\times\vec{A}) + \vec{\nabla}\phi\times\vec{A}$$

$$[\ \vec{B}\times\vec{A} = \hat{i}(B_yA_z - A_yB_z)$$
$$+ \hat{j}(B_zA_x - A_zB_x)$$
$$+ \hat{k}(B_xA_y - A_xB_y)$$

Prove that-

① $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

L.H.S $= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\{\hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z)$
$+ \hat{k}(A_xB_y - A_yB_x)\}$

$= \frac{\partial}{\partial x}(A_yB_z - A_zB_y) + \frac{\partial}{\partial y}(A_zB_x - A_xB_z) + \frac{\partial}{\partial z}(A_xB_y - A_yB_x)$

$= A_y\frac{\partial B_z}{\partial x} + \frac{\partial A_y}{\partial x}B_z - A_z\frac{\partial B_y}{\partial x} - \frac{\partial A_z}{\partial x}B_y + A_z\frac{\partial B_x}{\partial y} + \frac{\partial A_z}{\partial y}B_x$
$- A_x\frac{\partial B_z}{\partial y} - \frac{\partial A_x}{\partial y}B_z + A_x\frac{\partial B_y}{\partial z} + \frac{\partial A_x}{\partial z}B_y - A_y\frac{\partial B_x}{\partial z}$
$- \frac{\partial A_y}{\partial z}B_x$

$= B_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + B_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + B_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$
$- A_x\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - A_y\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)$
$- A_z\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$

$= (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z)\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} - (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & B_z \end{vmatrix}$

$= \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

= R.H.S //

(2) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$

L.H.S $= \vec{\nabla} \times \vec{\nabla}\phi$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial y}\right)$

$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$

$= \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)\right\}$
$+ \hat{k}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right\}$

$= i\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) + \hat{j}\left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right)$
$+ \hat{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$

$= 0 =$ R.H.S //

$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$ etc since x, y, z are independent variable.

(3) LHS $= \vec{\nabla} \cdot \vec{\nabla} \times \vec{A}$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$

$= \cancel{\frac{\partial^2 A_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_y}{\partial x \partial z}} + \cancel{\frac{\partial^2 A_x}{\partial y \partial z}} - \cancel{\frac{\partial^2 A_z}{\partial y \partial x}} + \cancel{\frac{\partial^2 A_y}{\partial z \partial x}} - \cancel{\frac{\partial^2 A_x}{\partial z \partial y}}$

$= 0$

$=$ R.H.S

④ $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$= grad\,(div\,\vec{A}) - \nabla^2 A$

L.H.S $= \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left\{\hat{i}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right\}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right\}$$
$$+ \hat{j}\left\{\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right\}$$

$$= \hat{i}\left(\frac{\partial^2 A_y}{\partial y\partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z\partial x}\right)$$
$$+ \hat{j}\left(\frac{\partial^2 A_z}{\partial z\partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x\partial y}\right)$$
$$+ \hat{k}\left\{\frac{\partial^2 A_x}{\partial x\partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y\partial z}\right\}$$

$$= \hat{i}\left(\frac{\partial^2 A_y}{\partial y\partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_z}{\partial z\partial x}\right)$$
$$+ \hat{j}\left(\frac{\partial^2 A_z}{\partial z\partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_x}{\partial x\partial y}\right)$$
$$+ \hat{k}\left(\frac{\partial^2 A_x}{\partial x\partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_z}{\partial z^2} + \frac{\partial^2 A_z}{\partial z^2} + \frac{\partial^2 A_y}{\partial y\partial z}\right)$$

$$= \left(-\frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2}\right)\hat{i} + \left(-\frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_y}{\partial y^2} - \frac{\partial^2 A_y}{\partial z^2}\right)\hat{j}$$
$$+ \left(-\frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_z}{\partial z^2}\right)\hat{k} + \hat{i}\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$
$$+ \hat{j}\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) + \nabla(\nabla\cdot A).$$

$= -\nabla^2 A + \nabla(\nabla\cdot A)$

$= \nabla(\nabla\cdot A) - \nabla^2 A$ = R.H.S //

Q. Find $\vec{\nabla}\phi$ if $\phi = \ln|\vec{r}|$

Sol: $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$|\vec{r}| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$

$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \ln(x^2+y^2+z^2)^{1/2}$

$= \hat{i}\frac{\partial}{\partial x}\left[\frac{1}{2}\ln(x^2+y^2+z^2)\right] + \hat{j}\frac{\partial}{\partial y}\left[\frac{1}{2}\ln(x^2+y^2+z^2)\right]$

$+ \hat{k}\frac{\partial}{\partial z}\left[\frac{1}{2}\ln(x^2+y^2+z^2)\right]$

$= \hat{i}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{x^2+y^2+z^2} + \hat{j}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}y}{x^2+y^2+z^2} + \hat{k}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}z}{x^2+y^2+z^2}$

$= \dfrac{\hat{i}x + \hat{j}y + \hat{k}z}{x^2+y^2+z^2}$

$= \dfrac{\vec{r}}{r^2}$

Q. Find $\vec{\nabla}\phi$ if $\phi = \frac{1}{r}$

Sol: $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

$\vec{\nabla}\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$

$= \hat{i}\left(-\frac{1}{\cancel{2}}\right)(x^2+y^2+z^2)^{-3/2} \times \cancel{2}x + \hat{j}\left(-\frac{1}{\cancel{2}}\right)(x^2+y^2+z^2)^{-3/2} \times \cancel{2}y$

$+ \hat{k}\left(-\frac{1}{\cancel{2}}\right)(x^2+y^2+z^2)^{-3/2} \times \cancel{2}z$

$= \dfrac{-(\hat{i}x + \hat{j}y + \hat{k}z)}{(x^2+y^2+z^2)^{3/2}}$

$= \dfrac{-\vec{r}}{r^3}$

Q. Find $\vec{\nabla}\phi$ if $\phi = r^n$

$\Rightarrow$ $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

$\therefore r^n = (x^2+y^2+z^2)^{n/2}$

$$\nabla(r^n) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \hat{i}.\frac{n}{2}.2x(x^2+y^2+z^2)^{n/2-1} + \hat{j}.\frac{n}{2}.2y(x^2+y^2+z^2)^{n/2-1}$$
$$+\hat{k}.\frac{n}{2}.(x^2+y^2+z^2)^{n/2-1}.2z$$

$$= n(\hat{i}x+\hat{j}y+\hat{k}z)(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= n\vec{r}\, r^{2\times\frac{n-2}{2}}$$

$$= n\vec{r}\, r^{n-2}$$

$$= n r^{n-2}\vec{r}$$

Q. Find $\vec{\nabla}|\vec{r}|^3$

Soln: $|\vec{r}| = (x^2+y^2+z^2)^{1/2}$

$$\vec{\nabla}|\vec{r}|^3 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{3/2}$$

$$= \hat{i}.\frac{3}{2}.2x(x^2+y^2+z^2)^{3/2-1} + \hat{j}.\frac{3}{2}.2y(x^2+y^2+z^2)^{3/2-1}$$
$$+\hat{k}.\frac{3}{2}.2z(x^2+y^2+z^2)^{3/2-1}$$

$$= (\hat{i}.3x + \hat{j}.3y + \hat{k}\,3z)\, r^{2\times\frac{1}{2}}$$

$$= 3\vec{r}\, r$$

Q. Evaluate $\vec{\nabla}\left(3r^2 - 4\sqrt{r} + \frac{6}{\sqrt[3]{r}}\right)$

Soln:

$$= 3.2\vec{r} - 4.\frac{1}{2}r^{\frac{1}{2}-2}\vec{r} + 6\times(-\tfrac{1}{3})\, r^{-\frac{1}{3}-2}\vec{r}$$

$$= 6\vec{r} - 2r^{-\frac{3}{2}}\vec{r} - 2r^{-\frac{7}{3}}\vec{r}$$

$$= 6\vec{r} - 2\frac{\vec{r}}{\sqrt{r^3}} - 2\frac{\vec{r}}{r^{7/3}}$$

Q. $\phi = 2xz^4 - x^2y$. Find $\vec{\nabla}\phi$ and $|\vec{\nabla}\phi|$ at the point $(2,-2,-1)$

$$\Rightarrow \vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2xz^4 - x^2y)$$

$$= \hat{i}\frac{\partial}{\partial x}(2xz^4 - x^2y) + \hat{j}\frac{\partial}{\partial y}(2xz^4 - x^2y) + \hat{k}\frac{\partial}{\partial z}(2xz^4 - x^2y)$$

$$= \hat{i}(2z^4 - 2xy) + \hat{j}(-x^2) + \hat{k}(2x.4z^3)$$

$= \hat{i}\,(2z^4 - 2xy) - \hat{j}\,x^2 + 8\hat{k}\,xz^3$

$|\nabla\phi| = \sqrt{(2z^4-2xy)^2 + x^4 + (8xz^3)^2}$

$= \sqrt{\{2\cdot(-1)^4 - 2\cdot(2)(-2)\}^2 + 2^4 + \{8\cdot 2\,(-1)^3\}^2}$

$= \sqrt{(2+8)^2 + 16 + (-16)^2}$

$= \sqrt{100 + 16 + 256}$

$= \sqrt{\cancel{308}\ 372}$

$= 2\sqrt{93}$ //

$256 + 52 = 308$

$256 + 122 = 378$

15/08

1) Let U be a differentiable funcn of x, y & z. Prove that $\vec{\nabla}U.d\vec{r} = dU$

Now, $dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$

$= (\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$

$= \vec{\nabla}U.d\vec{r}$

2) Let $\vec{\nabla}U = 2r^4\vec{r}$ Find U

=) we know that $\vec{\nabla}U.d\vec{r} = dU$

Integrating both sides :

$\int dU = \int \vec{\nabla}U.d\vec{r}$

$\Rightarrow U = \int 2r^4\vec{r}.d\vec{r} + c$

$= \int 2r^5 dr + c$

$= 2\frac{r^6}{6} + c$

$U = \frac{r^6}{3} + c$

③ Find $\phi(r)$ such that $\vec{\nabla}\phi = \frac{\vec{r}}{r^5}$ and $\phi(1)=0$

=)

$\underline{dU = \vec{\nabla}U . d\vec{r}}$. $d\phi = \vec{\nabla}\phi . d\vec{r}$

Integrating $U = \int \vec{\nabla}U . d\vec{r} + c$

$= \int$

$\phi = \int \vec{\nabla}\phi . d\vec{r}$

$= \int \frac{\vec{r}}{r^5} . d\vec{r} + c$

$\phi(r) = \int \frac{dr}{r^4} + c$

$= -\frac{1}{3r^3} + c$

At $r=1$, $\phi(1) = 0$

$\therefore \; 0 = -\frac{1}{3} + c$

$\Rightarrow c = \frac{1}{3}$

$\therefore \phi(r) = \frac{1}{3} - \frac{1}{3r^3}$

$= \frac{1}{3}(1 - \frac{1}{r^3})$

④ Prove $\vec{\nabla} f(r) = \frac{f'(r)\vec{r}}{r}$

Pf. $\vec{\nabla} f(r) = \hat{i}\frac{\partial}{\partial x} f(r) + \hat{j}\frac{\partial f(r)}{\partial y} + \hat{k}\frac{\partial}{\partial z} f(r)$

$= \hat{i}\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}$

$= \frac{\partial f}{\partial r}\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$

$= \frac{\partial f}{\partial r}\vec{\nabla} r$

Now $\vec{\nabla} r^n = n r^{n-2}\vec{r}$

$\vec{\nabla} r = 1 . r^{1-2}\vec{r}$

$= \frac{\vec{r}}{r}$

$\vec{\nabla} f(r) = \frac{\partial f}{\partial r}\frac{\vec{r}}{r} = \frac{f'(r)\vec{r}}{r}$

⑤ Suppose, F is a differentiable funcⁿ of x, y, z and t where x, y, z are differentiable funcⁿs of t. Prove that $\frac{dF}{dt} = \frac{\partial F}{\partial t} + \vec{\nabla} F \cdot \frac{d\vec{r}}{dt}$

$\Rightarrow$ $$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial t}dt$$

$$\therefore \frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} + \frac{\partial F}{\partial t}$$

$$\therefore \frac{dF}{dt} = \frac{\partial F}{\partial t} + \left(\hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}\right) \cdot \left(\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}\right)$$

$$= \frac{\partial F}{\partial t} + \vec{\nabla} F \cdot \frac{d}{dt}(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \frac{\partial F}{\partial t} + \vec{\nabla} F \cdot \frac{d\vec{r}}{dt} \quad //$$

⑥ Let $\vec{A}$ be a constant vector Prove that $\vec{\nabla}(\vec{r} \cdot \vec{A}) = \vec{A}$

$\Rightarrow$ $$\vec{r} \cdot \vec{A} = (\hat{i}x + \hat{j}y + \hat{k}z) \cdot (\hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3)$$

$$= xA_1 + yA_2 + zA_3$$

where A_1, A_2 & A_3 are constants.

$$\vec{\nabla}(\vec{r} \cdot \vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xA_1 + yA_2 + zA_3)$$

$$= \hat{i}\frac{\partial}{\partial x}(xA_1 + yA_2 + zA_3) + \hat{j}\frac{\partial}{\partial y}(xA_1 + yA_2 + zA_3) + \hat{k}\frac{\partial}{\partial z}(xA_1 + yA_2 + zA_3)$$

$$= \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$$

$$= \vec{A}$$

⑦ $$\vec{\nabla}\left(\frac{F}{G}\right) = \frac{G\vec{\nabla}F - F\vec{\nabla}G}{G^2} \quad \text{if } G \neq 0$$

$\Rightarrow$ $$\vec{\nabla}(F/G) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(F/G)$$

$$= \hat{i}\left(\frac{G\frac{\partial F}{\partial x} - F\frac{\partial G}{\partial x}}{G^2}\right) + \hat{j}\left(\frac{G\frac{\partial F}{\partial y} - F\frac{\partial G}{\partial y}}{G^2}\right) + \hat{k}\left(\frac{G\frac{\partial F}{\partial z} - F\frac{\partial G}{\partial z}}{G^2}\right)$$

$$= \frac{G\left(\hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}\right) - F\left(\hat{i}\frac{\partial G}{\partial x} + \hat{j}\frac{\partial G}{\partial y} + \hat{k}\frac{\partial G}{\partial z}\right)}{G^2}$$

$$= \frac{G\vec{\nabla}F - F\vec{\nabla}G}{G^2}$$

① Prove that $\nabla^2\left(\frac{1}{r}\right)=0$

$$\nabla^2\left(\frac{1}{r}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(x^2+y^2+z^2)^{-1/2}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-1/2}=\frac{\partial}{\partial x}\left[-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\times 2x\right]$$

$$=-(x^2+y^2+z^2)^{-3/2}+x.\frac{3}{2}(x^2+y^2+z^2)^{-5/2}.2x$$

$$=-r^{-3}+3x^2r^{-5}$$

$$\frac{\partial^2}{\partial y^2}(x^2+y^2+z^2)^{-1/2}=-r^{-3}+3y^2r^{-5}$$

$$\frac{\partial^2}{\partial z^2}(x^2+y^2+z^2)^{-1/2}=-r^{-3}+3z^2r^{-5}$$

$$\nabla^2\left(\frac{1}{r}\right)=-3r^{-3}+3(x^2+y^2+z^2)r^{-5}$$

$$=-\frac{3}{r^3}+\frac{3r^2}{r^2r^3}$$

$$=-\frac{3}{r^3}+\frac{3}{r^3}$$

$$=0$$

$\therefore$ a soln of the eqn $\nabla^2\phi=0$ is $\phi=\frac{1}{r}$

② Prove $\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right)=0$

Now, $\vec{\nabla}.(\phi\vec{A})=\vec{\nabla}\phi.\vec{A}+\phi(\vec{\nabla}.\vec{A})$

Putting $\phi=\frac{1}{r^3}$ and $\vec{A}=\vec{r}$

$$\vec{\nabla}.\left(\frac{\vec{r}}{r^3}\right)=\vec{\nabla}\left(\frac{1}{r^3}\right).\vec{r}+\frac{1}{r^3}(\vec{\nabla}.\vec{r})$$

Now $\vec{\nabla}.\vec{r}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(\hat{i}x+\hat{j}y+\hat{k}z)$

$$=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}$$

$$=3$$

$$\therefore\ \vec{\nabla}.\vec{r}=3$$

Again $\vec{\nabla}r^n=nr^{n-2}\vec{r}$

$$\therefore\ \vec{\nabla}\left(\frac{1}{r^3}\right)=-3r^{-3-2}\vec{r}$$

$$=-3r^{-5}\vec{r}$$

$\therefore \quad \nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = -3r^{-5}\vec{r}\cdot\vec{r} + \frac{3}{r^3}$

$= -\frac{3r^2}{r^5} + \frac{3}{r^3}$

$= -\frac{3}{r^3} + \frac{3}{r^3}$

$= 0$ //

Q. Prove $\vec{\nabla}\cdot(U\vec{\nabla}V - V\vec{\nabla}U) = U\nabla^2 V - V\nabla^2 U$

$\vec{\nabla}\cdot(U\nabla V) = \vec{\nabla}U\cdot\vec{\nabla}V + U(\vec{\nabla}\cdot\vec{\nabla}V)$

$= \vec{\nabla}U\cdot\vec{\nabla}V + U\nabla^2 V$

$-\vec{\nabla}\cdot(V\vec{\nabla}U) = -\vec{\nabla}V\cdot\vec{\nabla}U - V\vec{\nabla}\cdot\vec{\nabla}U$

$= -\vec{\nabla}U\cdot\vec{\nabla}V - V\nabla^2 U$

$\therefore \quad \vec{\nabla}\cdot(U\vec{\nabla}V - V\nabla U) = U\nabla^2 V - V\nabla^2 U$ //

Note:-

Q. A vector whose divergence is zero is called a solenoidal vector. If $\vec{A}$ is a solenoidal vector then $\vec{\nabla}\cdot\vec{A} = 0$

Q. Determine the constant a so that the vector

$\vec{V} = (-4x - 6y + 3z)\hat{i} + (-2x + y - 5z)\hat{j} + (5x + 6y + az)\hat{k}$ is Solenoidal.

Soln If $\vec{V}$ is a solenoidal vector $\vec{\nabla}\cdot\vec{V} = 0$

i.e $\frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y + \frac{\partial}{\partial z}V_z = 0$

$\Rightarrow \frac{\partial}{\partial x}(-4x - 6y + 3z) + \frac{\partial}{\partial y}(-2x + y - 5z) + \frac{\partial}{\partial z}(5x + 6y + az) = 0$

$\Rightarrow -4 + 1 + a = 0$

$\Rightarrow a = 3$

$\therefore \vec{\nabla}\cdot\vec{V} = 0$ if $a = 3$ //

8) Prove: $\nabla^2 r^n = n(n+1) r^{n-2}$ where n is a constant.

$$\nabla r^n = n r^{n-2} \vec{r}$$

$$\nabla^2 r^n = \vec{\nabla} \cdot \nabla r^n$$

$$= \vec{\nabla} \cdot (n r^{n-2} \vec{r})$$

$$= n \vec{\nabla} \cdot (r^{n-2} \vec{r})$$

$$= n [\vec{\nabla} r^{n-2} \cdot \vec{r} + r^{n-2} \vec{\nabla} \cdot \vec{r}]$$

$$= n(n-2) r^{n-2-2} \vec{r} \cdot \vec{r} + 3 r^{n-2} n$$

$$= n(n-2) r^{n-4} r^2 + 3 r^{n-2} n$$

$$= n(n-2) r^{n-2} + 3 r^{n-2} n$$

$$= [3n + n(n-2)] r^{n-2}$$

$$= [3n + n^2 - 2n] r^{n-2}$$

$$= [n^2 + n] r^{n-2}$$

$$= n(n+1) r^{n-2}$$

H.W ① Evaluate $\mathrm{div}(2x^2 z \hat{i} - x y^2 z \hat{j} + 3 y z^2 \hat{k})$

② Let $\phi = 3x^2 z - y^2 z^3 + 4x^3 y + 2x - 3y - 5$

find $\nabla^2 \phi$

① $\vec{\nabla} \cdot (2x^2 z \hat{i} - x y^2 z \hat{j} + 3 y z^2 \hat{k})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x^2 z \hat{i} - x y^2 z \hat{j} + 3 y z^2 \hat{k})$$

$$= \frac{\partial}{\partial x}(2x^2 z) - \frac{\partial}{\partial y}(x y^2 z) + \frac{\partial}{\partial z}(3 y z^2)$$

$$= 4xz - 2xyz + 6yz$$

③ $\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5)$

$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}3x^2z\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}4x^3y\right) + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}2x\right)$

$+ \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(-y^2z^3)\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}4x^3y\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(-3y)\right)$

$+\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}3x^2z\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}(-y^2z^3)\right).$

$= \frac{\partial}{\partial x}(6xz) + \frac{\partial}{\partial x}(12x^2y) - \frac{\partial}{\partial y}(2yz^3) + \frac{\partial}{\partial y}(4x^3) - \frac{\partial}{\partial y}(3)$

$+ \frac{\partial}{\partial z}(3x^2) + \frac{\partial}{\partial z}(y^2.3z^2)$

$= 6z + 24xy - 2z^3 - 6y^2z$ //

Q. Evaluate $\nabla^2(\ln r)$

=) $\nabla^2(\ln r) = \vec{\nabla}.\vec{\nabla}(\ln r)$

$=\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\frac{\vec{r}}{r^2}$

$= \vec{\nabla}.\left(\frac{\vec{r}}{r^2}\right)$

$= \frac{1}{r^2}\vec{\nabla}.\vec{r} + \vec{\nabla}\left(\frac{1}{r^2}\right).\vec{r}$

$= \frac{3}{r^2} - 2r^{-4}\,\vec{r}.\vec{r}$

$= \frac{3}{r^2} - \frac{2\vec{r}^2}{r^4}$

$= \frac{3}{r^2} - \frac{2}{r^2}$

$= \frac{1}{r^2}$

Q. Suppose $\vec{\omega}$ is a constant vector and $\vec{v} = \vec{\omega}\times\vec{r}$.
Prove that $\text{div}\,\vec{v} = 0$

$\vec{\nabla}.\vec{v} = \vec{\nabla}.(\vec{\omega}\times\vec{r})$

$= \vec{r}.(\vec{\nabla}\times\vec{\omega}) - \vec{\omega}.(\vec{\nabla}\times\vec{r})$

$$\vec{\nabla} \times \vec{\omega} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

$\vec{\omega}$ is constant, hence $\omega_1, \omega_2, \omega_3$ are constants i.e $\vec{\nabla} \times \vec{\omega} = 0$

$$\vec{\nabla} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$\therefore \vec{\nabla} \cdot \vec{\nabla} = 0$

Q. Evaluate $\vec{\nabla} \cdot (r^3 \vec{r})$

$\Rightarrow \vec{\nabla} \cdot (r^3 \vec{r})$

$= \vec{\nabla} r^3 \cdot \vec{r} + r^3 \vec{\nabla} \cdot \vec{r}$

$= 3r^{3-2} \vec{r} \cdot \vec{r} + r^3 \times 3$

$= 3r r^2 + 3r^3$

$= 3r^3 + 3r^3$

$= 6r^3$

Q. Evaluate $\vec{\nabla} \cdot \left[r \vec{\nabla}\left(\frac{1}{r^3}\right) \right]$

$\Rightarrow \vec{\nabla} \cdot \left[r \vec{\nabla}\left(\frac{1}{r^3}\right) \right]$

$= \vec{\nabla} r \cdot \vec{\nabla}\left(\frac{1}{r^3}\right) + r \vec{\nabla} \cdot \vec{\nabla}\left(\frac{1}{r^3}\right)$

$= \frac{\vec{r}}{r} \cdot (-3r^{-5}\vec{r}) + r\vec{\nabla} \cdot (-3r^{-5}\vec{r})$

$= -\frac{3\vec{r} \cdot \vec{r}}{r^6} - 3r^{-5} r \vec{\nabla} \cdot \vec{r}$

$= -\frac{3}{r^4} - 3 \cdot 3 \frac{1}{r^4}$

$= -\frac{12}{r^4}$

$\vec{\nabla} r$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2}$

$= \hat{x}\frac{1}{2}(x^2+y^2+z^2)^{-1/2} \times 2x$

$+ \hat{j}\frac{1}{2}(x^2+y^2+z^2)^{-1/2} \times 2y$

$+ \hat{k}\frac{1}{2}(x^2+y^2+z^2)^{-1/2} \times z$

$= \frac{1}{r}(\hat{i}x + \hat{j}y + \hat{k}z)$

$= \frac{\vec{r}}{r}$

$$\frac{\vec{r}}{r}\cdot(-3r^{-5}\vec{r}) + r\,\nabla^2\left(\frac{1}{r^3}\right)$$

$$= -3\frac{\vec{r}\cdot\vec{r}}{r^6} + r(-3)(-3+1)r^{-3-2} \qquad \nabla^2 r^n = n(n+1)r^{n-2}$$

$$= -\frac{3}{r^4} + \frac{6}{r^4}$$

$$= \frac{3}{r^4}$$

Q) If $\vec{A} = \frac{\vec{r}}{r}$, find grad div $\vec{A}$ $\;\; \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

$$= \vec{\nabla}\left(\frac{2}{r}\right)$$

$$= 2\vec{\nabla}\left(\frac{1}{r}\right)$$

$$= 2\times\left(-\frac{\vec{r}}{r^3}\right)$$

$$= -\frac{2\vec{r}}{r^3}$$

Now $\vec{\nabla}\cdot\frac{\vec{r}}{r}$

$$= \frac{1}{r}\vec{\nabla}\cdot\vec{r} + \vec{\nabla}\left(\frac{1}{r}\right)\cdot\vec{r}$$

$$= \frac{3}{r} - \frac{\vec{r}}{r^3}\cdot\vec{r}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$

Q) Evaluate $\nabla^2\left[\vec{\nabla}\cdot\frac{\vec{r}}{r^2}\right]$

$$\nabla^2\left[\frac{1}{r^2}\vec{\nabla}\cdot\vec{r} + \vec{\nabla}\left(\frac{1}{r^2}\right)\cdot\vec{r}\right]$$

$$= \nabla^2\left[\frac{3}{r^2} - 2r^{-2-2}\,\vec{r}\cdot\vec{r}\right]$$

$$= \nabla^2\left[\frac{3}{r^2} - \frac{2r^2}{r^4}\right]$$

$$= \nabla^2\left[\frac{1}{r^2}\right]$$

$$\left[\nabla^2 r^n = n(n+1)r^{n-2}\right]$$

$$= (-2)(-2+1)r^{-2-2}$$

$$= (-2\times -1)r^{-4}$$

$$= \frac{2}{r^4}$$

//

Q. a) Prove $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$

b) Find $f(r)$ such that $\nabla^2 f(r) = 0$

$[\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z]$

$$\nabla^2 f(r) = \vec{\nabla} \cdot \vec{\nabla} f(r)$$

$$= \vec{\nabla} \cdot \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(r)$$

$$= \vec{\nabla} \cdot \left(\hat{i}\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial z}\right)$$

$$= \vec{\nabla} \cdot \left(\hat{i}\frac{\partial f}{\partial r} \times \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}\cancel{2}x + \hat{j}\frac{\partial f}{\partial r}\frac{1}{\cancel{2}}\cancel{2}y(x^2+y^2+z^2)^{-1/2} + \hat{k}\frac{\partial f}{\partial r}\frac{1}{\cancel{2}}\cancel{2}z(x^2+y^2+z^2)^{-1/2}\right)$$

$$= \vec{\nabla} \cdot (\hat{i}x + \hat{j}y + \hat{k}z)\frac{1}{r}\frac{\partial f}{\partial r}$$

$$= \vec{\nabla} \cdot \left(\frac{\vec{r}}{r}\frac{\partial f}{\partial r}\right)$$

$$= \frac{1}{r}\frac{\partial f}{\partial r}\vec{\nabla}\cdot\vec{r} + \vec{\nabla}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right)\cdot\vec{r}$$

$$= \frac{3}{r}\frac{\partial f}{\partial r}$$

$$\vec{\nabla}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{1}{r}\frac{\partial f}{\partial r}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial}{\partial x}\frac{\partial f}{\partial r}$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right)\frac{\partial r}{\partial x}$$

$$= -\frac{1}{r^2}\cdot\frac{x}{r}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial r^2}\cdot\frac{x}{r}$$

$$\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right) = -\frac{1}{r^2}\hat{i}\frac{x}{r}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial r^2}\frac{\hat{i}x}{r}$$

$$\hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right) = -\frac{1}{r^2}\hat{j}\frac{y}{r}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial r^2}\frac{\hat{j}y}{r}$$

$$\hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right) = -\frac{1}{r^2}\frac{\hat{k}z}{r}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial r^2}\frac{\hat{k}z}{r}$$

$$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$$

$$\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(r)\right] = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial r} \times \frac{\partial r}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial r}\cdot\frac{x}{r}\right]$$

$$= \frac{\partial}{\partial x}(x)\frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right)x$$

$$= \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right)\frac{\partial r}{\partial x}\cdot x$$

$$= \frac{1}{r}\frac{\partial f}{\partial r} + \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial f}{\partial r}\right]\frac{\partial r}{\partial x}\cdot x + \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right)\times\frac{1}{r}\frac{\partial r}{\partial x}\cdot x$$

$$= \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}\frac{\partial f}{\partial r}\frac{x^2}{r} + \frac{\partial^2 f}{\partial r^2}\frac{x^2}{r^2}$$

$$\frac{\partial^2}{\partial x^2}[f(r)] = \frac{1}{r}\frac{\partial f}{\partial r} - \frac{x^2}{r^3}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\frac{x^2}{r^2}$$

$$\frac{\partial^2}{\partial y^2}[f(r)] = \frac{1}{r}\frac{\partial f}{\partial r} - \frac{y^2}{r^3}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\frac{y^2}{r^2}$$

$$\frac{\partial^2}{\partial z^2}[f(r)] = \frac{1}{r}\frac{\partial f}{\partial r} - \frac{z^2}{r^3}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\frac{z^2}{r^2}$$

$$\nabla^2 f(r) = \frac{3}{r}\frac{\partial f}{\partial r} - \frac{(x^2+y^2+z^2)}{r^3}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}\frac{(x^2+y^2+z^2)}{r^2}$$

$$= \frac{3}{r}\frac{\partial f}{\partial r} - \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} = \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}$$

(b) $\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} = 0$

$\nabla^2 f(r) = 0$

we know $\nabla^2\left(\frac{1}{r}\right) = 0$

Let $f(r) = A + \frac{B}{r}$

$$\vec{\nabla}\cdot\vec{\nabla} f(r) = \vec{\nabla}\cdot\vec{\nabla}\left(A + \frac{B}{r}\right)$$

$$= \vec{\nabla}\cdot\vec{\nabla}A + \vec{\nabla}\cdot\vec{\nabla}\left(\frac{B}{r}\right)$$

$$= 0 + B\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right)$$

$$= B\nabla^2\left(\frac{1}{r}\right)$$

$$= 0$$

Q) Evaluate $\vec{\nabla}\cdot\vec{A}\times\vec{r}$ if $\vec{\nabla}\times\vec{A} = 0$

$$\vec{\nabla}\cdot\vec{A}\times\vec{B} = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}$$

$$\vec{\nabla}\cdot\vec{A}\times\vec{r} = \vec{r}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{r}$$

$$= -\vec{A}\cdot\vec{\nabla}\times\vec{r})$$

$$= 0$$

Q. Find curl$\{\vec{r} f(\vec{r})\}$ where $f(\vec{r})$ is differentiable

$\Rightarrow$ curl$\{\vec{r} f(r)\}$

$$= \vec{\nabla} \times (\vec{r} f(r))$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(z f(r)) - \frac{\partial}{\partial z}(y f(r))\right\} + \hat{j}\left\{\frac{\partial}{\partial z}(x f(r)) - \frac{\partial}{\partial x}(z f(r))\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(y f(r)) - \frac{\partial}{\partial y}(x f(r))\right\}$$

$$= \hat{i}\left\{z \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial z}\right\} + \hat{j}\left\{x \frac{\partial f}{\partial z} - z \frac{\partial f}{\partial x}\right\} + \hat{k}\left\{y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y}\right\} \quad \text{— (1)}$$

$$\frac{\partial f(r)}{\partial x} = \frac{\partial f(r)}{\partial r} \times \frac{\partial r}{\partial x}$$
$$= f'(r) \times \frac{1}{2} \frac{2x}{(x^2+y^2+z^2)^{1/2}}$$
$$= \frac{f'(r)\, x}{r}$$

S'ly $\quad \frac{\partial f(r)}{\partial y} = \frac{f'(r)\, y}{r}$

$$\frac{\partial f(r)}{\partial z} = \frac{f'(r)\, z}{r}$$

(1) $\Rightarrow$ $\hat{i}\left\{\frac{z f'(r) y}{r} - \frac{y z f'(r)}{r}\right\} + \hat{j}\left\{\frac{x z f'(r)}{r} - \frac{z x f'(r)}{r}\right\}$
$$+ \hat{k}\left\{\frac{y x f'(r)}{r} - \frac{x y f'(r)}{r}\right\}$$

$= 0$

Q. Suppose $\vec{v} = \vec{\omega} \times \vec{r}$, Prove that $\vec{\omega} = \frac{1}{2}$ curl $\vec{v}$ where $\vec{\omega}$ is a const. vector.

$\Rightarrow$ curl $\vec{v} = \vec{\nabla} \times \vec{v}$

$$= \vec{\nabla} \times (\vec{\omega} \times \vec{r})$$

$$= \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\nabla} \times \{\hat{i}(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z)\right\}$$
$$+ \hat{j}\left\{\frac{\partial}{\partial z}(\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x}(\omega_1 y - \omega_2 x)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y)\right\}$$

$$= \hat{i}(\omega_1 + \omega_1) + \hat{j}(\omega_2 + \omega_2) + \hat{k}(\omega_3 + \omega_3)$$

$$= 2(\hat{i}\omega_1 + \hat{j}\omega_2 + \hat{k}\omega_3)$$

$\therefore$ curl $\vec{v} = 2\vec{\omega}$

$\Rightarrow \vec{\omega} = \frac{1}{2}$ curl $\vec{v}$ //

Q) Suppose $\vec{\nabla}\cdot\vec{E} = 0$, $\vec{\nabla}\cdot\vec{H} = 0$ $\quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{H}}{\partial t}$, $\vec{\nabla}\times\vec{H} = \frac{\partial \vec{E}}{\partial t}$

Show that $\vec{E}$ & $\vec{H}$ satisfy $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{H}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\frac{\partial \vec{H}}{\partial t}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

$$\Rightarrow \quad +\nabla^2 E = +\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial t}\right)$$

$$\nabla^2 E = \frac{\partial^2 E}{\partial t^2}$$

Now, $\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \nabla\times\frac{\partial \vec{E}}{\partial t}$

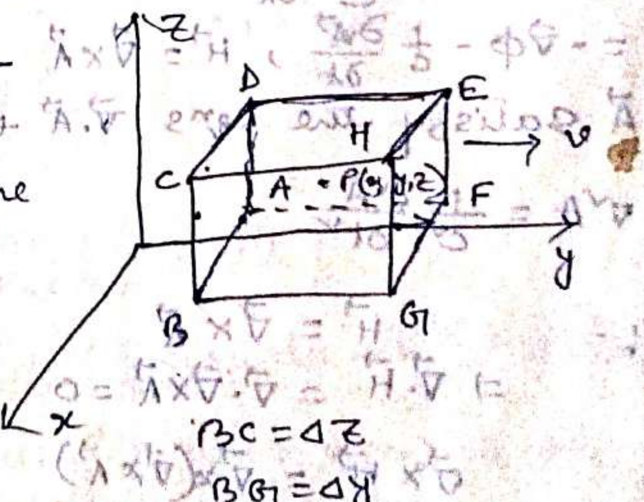
$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot H) - \nabla^2 H = \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$-\nabla^2 H = \frac{\partial}{\partial t}\left(-\frac{\partial \vec{H}}{\partial t}\right)$$

$$\Rightarrow \nabla^2 H = \frac{\partial^2 H}{\partial t^2}$$

A vector is said to be irrotational if curl $\vec{v} = 0$ & solenoidal if $\vec{\nabla}\cdot\vec{v} = 0$

Q) A fluid moves so that its velocity at any point is $\vec{v}(x,y,z)$. Show that the loss of fluid per unit volm per unit time in a small parallelopiped having centre at $P(x,y,z)$ and edges parallel to the coordinate axes and having magnitude $\Delta x, \Delta y, \Delta z$ respectively is given approximately by $\operatorname{div}\vec{v} = \vec{\nabla}\cdot\vec{v}$.



Soln: x component of velocity $\vec{v}$ at $P = v_x$

x component of velocity $\vec{v}$ at the centre of face ~~ABG~~ BGHC

$v_x + \frac{\partial v_x}{\partial x}\frac{\Delta x}{2}$ approx.

x component of velocity $\vec{v}$ at the centre of face AFED

$= v_x - \frac{\partial v_x}{\partial x}\frac{\Delta x}{2}$ approx.

$BC = \Delta z$

$BG = \Delta y$

$AB = \Delta x$

Hence ~~velocity~~ volm of fluid crossing ~~the volm ABCDEFGH~~ AFED per unit time $= \left(v_x - \frac{\partial v_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z$

vol^m of fluid crossing BGHC per unit time $= (V_x + \frac{\partial V_x}{\partial x}\frac{\Delta x}{2})\Delta y \Delta z$

vol^m of fluid flowing out per unit time from parallelopiped ABCDEFGH in the x direc^n $= (V_x + \frac{\partial V_x}{\partial x}\frac{\Delta x}{2} - V_x + \frac{\partial V_x}{\partial x}\frac{\Delta x}{2})\Delta y \Delta z$

$= \frac{\partial V_x}{\partial x}\Delta x \Delta y \Delta z$

Therefore vol^m of fluid flowing out from parallelopiped per unit time per unit vol^m in x direc^n $= \frac{\frac{\partial V_x}{\partial x}\Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$

$= \frac{\partial V_x}{\partial x}$

Similarly vol^m of fluid flowing out from parallelopiped per unit time per unit vol^m along y & z direc^n $\frac{\partial V_y}{\partial y}$ & $\frac{\partial V_z}{\partial z}$ respectively

∴ The loss of fluid per unit unit time per unit vol^m

$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$= \vec{\nabla}\cdot\vec{V}$

$= \operatorname{div}\vec{V}$

Q) The maxwell eq^n are $\vec{\nabla}\cdot\vec{H} = 0$, $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, $\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ &
$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$. Show that the soln to these eq^n are given by
$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, $\vec{H} = \vec{\nabla}\times\vec{A}$ where the scalar & vector potentials ϕ & $\vec{A}$ satisfy the eq^ns $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$, $\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho$ &

$\nabla^2 A = \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}$

Sol^n:- $\vec{H} = \vec{\nabla}\times\vec{A}$

$\Rightarrow \vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$

$\vec{\nabla}\times\vec{H} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$= \vec{\nabla}(-\frac{1}{c}\frac{\partial\phi}{\partial t}) - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$

$= -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi) - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\phi + \frac{1}{c}\frac{\partial A}{\partial t}\right)$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)$$

$$= \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)$$

$$= -\nabla^2\phi - \frac{1}{c}\nabla\cdot\frac{\partial \vec{A}}{\partial t}$$

$$= -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})$$

$$= -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \phi}{\partial t}\right)$$

$$= -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$= 4\pi\rho$$

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right)$$

$$= -\vec{\nabla}\times\vec{\nabla}\phi - \frac{1}{c}\left(\nabla\times\frac{\partial \vec{A}}{\partial t}\right)$$

$$= 0 - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{A})$$

$$= -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$$

Vector Integration :-

Line Integral,
Surface Integral,
Volume Integral

If $R(u) = \hat{i}R_1(u) + \hat{j}R_2(u) + \hat{k}R_3(u)$

then $\int \vec{R}(u)\,du = \hat{i}\int R_1(u)\,du + \hat{j}\int R_2(u)\,du + \hat{k}\int R_3(u)\,du$

Line Integral - If $\vec{r}(u) = \hat{i}x(u) + \hat{j}y(u) + \hat{k}z(u)$ is the position vector of the point $P(x, y, z)$ and if $\vec{r}(u)$ defines a curve joining points P_1 & P_2 where $u = u_1$ & $u = u_2$ respectively and if C is composed of a finite number of curves for each of which $\vec{r}(u)$ has a continuous derivative. Let $\vec{A}(x, y, z) = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$

be a vector funcⁿ of position defined and ~~con~~ continuous along C.

Then the integral of the tangential component of $\vec{A}$ along C from P_1 to P_2 is a line integral,

i.e $\int_{P_1}^{P_2} \vec{A}\cdot d\vec{r} = \int_C \vec{A}\cdot d\vec{r} = \int_C A_1 dx + A_2 dy + A_3 dz$

If C is a closed curve, the integral around C is written as

$\oint \vec{A}\cdot d\vec{r} = \oint A_1 dx + A_2 dy + A_3 dz$

① Find the total work done in moving a particle in the force field given by $\vec{F} = z\hat{i} + z\hat{j} + x\hat{k}$ along the helix C given by $x = \cos t$, $y = \sin t$, $z = t$ from $t = 0$ to $t = \frac{\pi}{2}$

Solⁿ:

$$W = \int \vec{F}\cdot d\vec{r} = \int_C (z\hat{i} + z\hat{j} + x\hat{k})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \int_C z\,dx + z\,dy + x\,dz$$

$dx = -\sin t\,dt$

$dy = \cos t\,dt$

$dz = dt$

$$W = \int_0^{\pi/2} -t\sin t\,dt + \int_0^{\pi/2} t\cos t\,dt + \int_0^{\pi/2} \cos t\,dt$$

$$= -\left[t(-\cos t) + \sin t\right]_0^{\pi/2} + \left[t\sin t + \cos t\right]_0^{\pi/2} + \left[\sin t\right]_0^{\pi/2}$$

$$= -\left[-t\cos t + \sin t\right]_0^{\pi/2} + \left[t\sin t + \cos t\right]_0^{\pi/2} + \left[\sin t\right]_0^{\pi/2}$$

$$= -\left[1\right] + \left[\frac{\pi}{2} - 1\right] + 1$$

$$= -1 + \frac{\pi}{2} - \cancel{1} + \cancel{1}$$

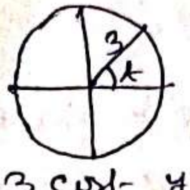
$$= \frac{\pi}{2} - 1$$

② Suppose a force field is given by,

$$\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}.$$

Find the work done in moving a particle once around a circle c in the x-y plane with its centre at the origin and a radius of 3. (ans 18π)

Soln: $W = \oint \vec{F}\cdot d\vec{r}$



$\vec{r} = x\hat{i} + y\hat{j}$
$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$

The parametric eqn's are $x = 3\cos t$ $y = 3\sin t$ and for work done along c once limits of are from $t = 0$ to $t = 2\pi$

$\vec{r} = 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$ $dx = -3\sin t$
$dy = 3\cos t$

$$W = \int_0^{2\pi} \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} (2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_0^{2\pi} (2x - y)\,dx + \int_0^{2\pi} (x + y)\,dy$$

$$= \int_0^{2\pi} 2x\,dx + \int_0^{2\pi} y\,dy$$

$$= 2\,\frac{x^2}{2}\;.\;2$$

$$W = \int_0^{2\pi} \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} (2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_0^{2\pi} (2x - y)\,dx + \int_0^{2\pi} (x + y)\,dy$$

$$= \int_0^{2\pi} (6\cos t - 3\sin t)(-3\sin t\,dt) + \int_0^{2\pi} (3\cos t + 3\sin t)\,3\cos t\,dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt$$

$$= 9\int_0^{2\pi} dt - 9\int_0^{2\pi} \sin t\cos t\,dt$$

$$= 9\cdot 2\pi - \frac{9}{2}\int_0^{2\pi} \sin 2t \, dt$$

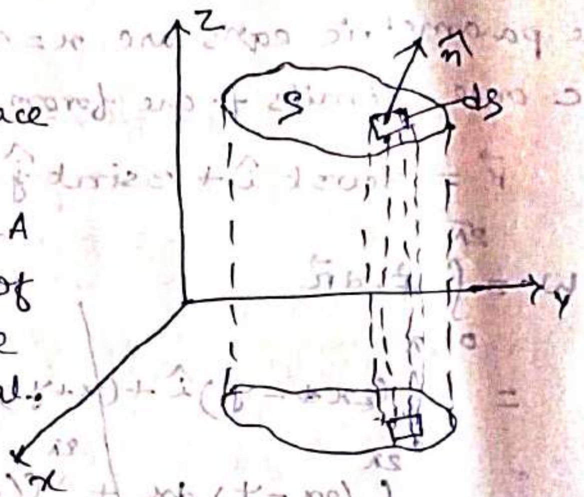
$$= 18\pi - \frac{9}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= 18\pi + \frac{9}{4}\left[\cos 2t\right]_{2\pi}^{0}$$

$$= 18\pi + \frac{9}{4}(1 - \cos 4\pi)$$

$$= 18\pi$$

Surface Integrals :-



Let S be a two sided surface. Let one side of S be considered arbitrarily as the positive side. A unit normal $\hat{n}$ to any point of the +ve side of S is called a +ve or outward drawn unit normal.

If $d\vec{s}$ is a vector whose magnitude is the differential area ds and whose direction is given by $\hat{n}$ then $d\vec{s} = \hat{n}\, ds$

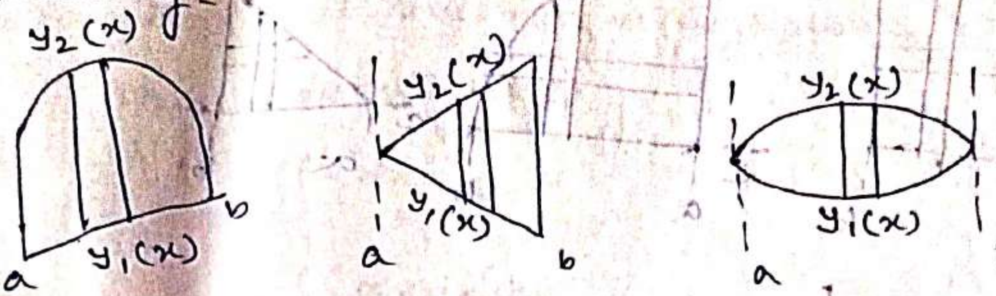
The integral $\iint \vec{A}\cdot d\vec{s} = \iint \vec{A}\cdot \hat{n}\, ds$ is a surface integral called the flux of $\vec{A}$ over the surface S.

Other surface integrals are $\iint_S \phi\, ds$, $\iint_S \phi\, \hat{n}\, ds$, $\iint_S \vec{A}\times d\vec{s}$

Volume Integrals :-

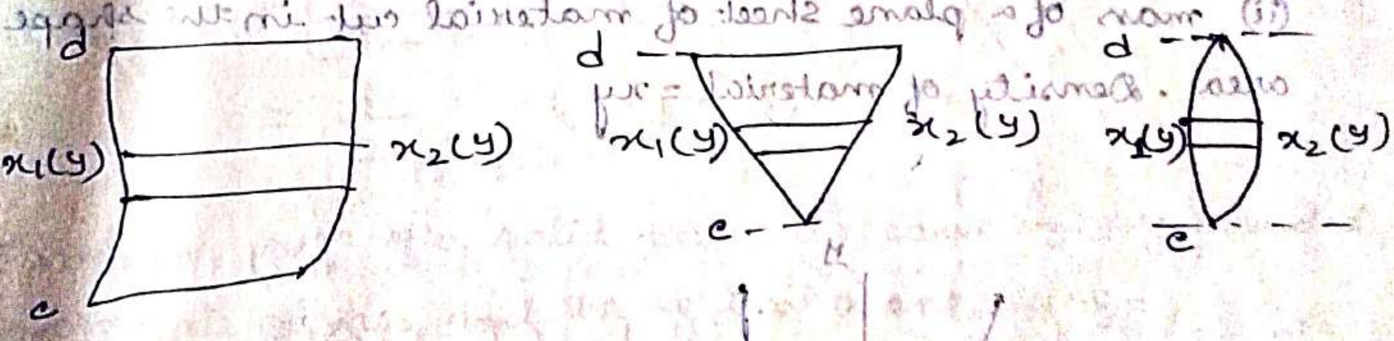
Let S be a closed surface in space enclosing a vol^m V. Then $\iiint_V \vec{A}\, dv$ & $\iiint_V \phi\, dv$ are called vol^m or space integral.

There are two methods of evaluating double integrals. Consider the following -



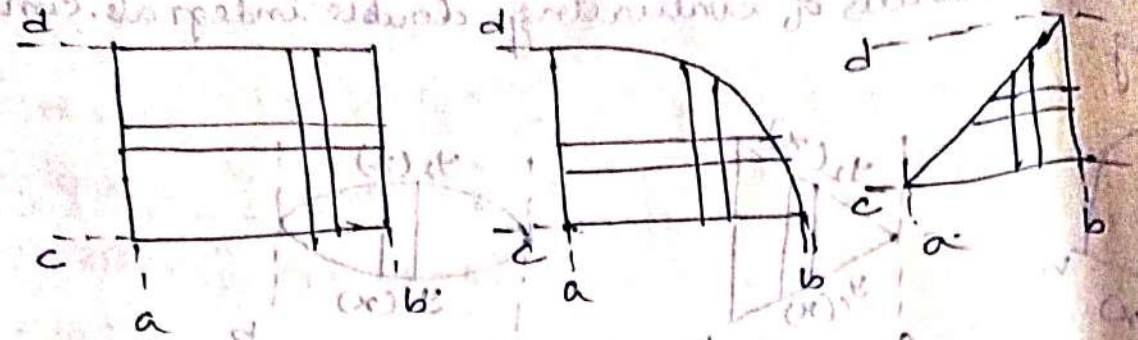
The top and bottom of these figures are areas bounded by known curves $y_1(x)$ & $y_2(x)$ with the boundaries at $x=a$ & $x=b$, which are either vertical or ~~horizental~~ lines or else points. In such cases, the integration over y is done first and then the integral is integrated over x within the limits $x=a$ & $x=b$.

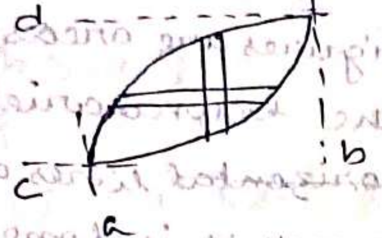
$$\iint f(x,y)\,dx\,dy = \int_{x=a}^{x=b}\left[\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\,dy\right]dx$$



The areas bounded by known curves $x_1(y)$ & $x_2(y)$ with boundaries at $y=c$ & $y=d$ which are either horizental, straight lines or else points. In such cases, the integration over x is done first & then the integral is integrated over y within the limits $y=c$ & $y=d$

$$\iint f(x,y)\,dx\,dy = \int_{y=c}^{y=d}\left[\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\right]dy$$





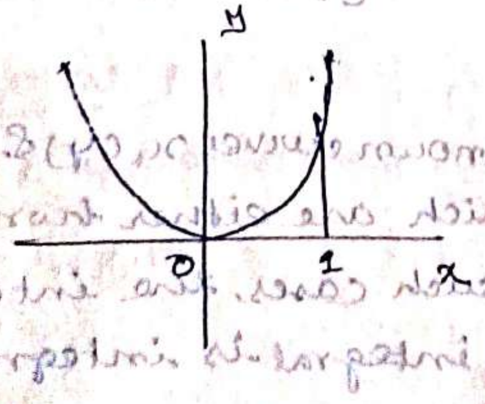
For a double Integral over a rectangle when $f(x,y)$ is a product of $(x,y) = g(x)h(y)$.

then $\int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy = \int_{x=a}^{b} g(x)\,dx * \int_{y=c}^{d} h(y)\,dy$

① Given a curve $y=x^2$ from $x=0$ to $x=1$ find

(i) area under curve.

(ii) mass of a plane sheet of material cut in the shape of the area. Density of material $= xy$



$A = \int_0^1 y\,dx$

$= \int_0^1 x^2\,dx$

$= \frac{1}{3}$

Mass = volm × density

For a plane sheet thickness is negligible.

$\therefore$ Mass = density × area

$$= \iint \rho \, dx\, dy$$

$$= \iint xy \, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{x^2} xy \, dx\, dy$$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{x^2} xy\, dy \right] dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} \right]_0^{x^2} dx$$

$$= \int_0^1 \frac{x \times x^4}{2} dx$$

$$= \int_0^1 \frac{x^5}{2} dx$$

$$= \left[\frac{x^6}{2 \times 6} \right]_0^1$$

$$= \frac{1}{12}$$

Q) 2. Find the volm of a solid below the curve $z = 1+y$ bounded by co-ordinate planes and the vertical plane $2x+y=2$

$$V = \int_0^1 \int_0^{2-2x} \int_{z=0}^{1+y} dx\, dy\, dz$$

$$= \int_0^1 \int_0^{y=2-2x} \left[\int_0^{1+y} dz \right] dy\, dx$$

$$= \int_0^1 \left[\int_0^{2-2x} (1+y)\, dy \right] dx = \int_0^1 \left[y + \frac{y^2}{2} \right]_0^{2-2x} dx$$

$$= \int_0^1 \left[2-2x + \frac{(2-2x)^2}{2} \right] dx$$

$$= \int_0^1 \left[2-2x+2(1-x)^2\right] dx$$

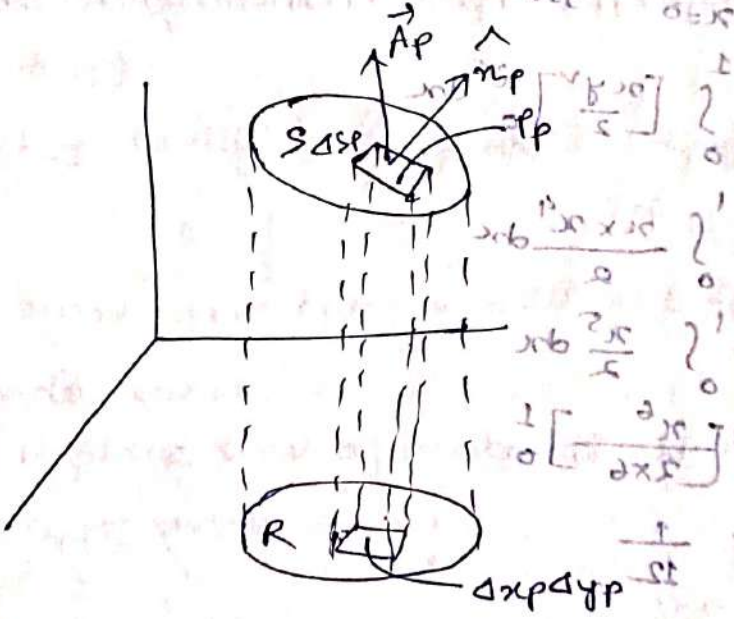
$$= \int_0^1 \left\{2-2x+2(1-2x+x^2)\right\} dx$$

$$= \left[4x - 6\cdot\frac{x^2}{2} + 2\cdot\frac{x^3}{3}\right]_0^1$$

$$= 4 - 3 + \frac{2}{3}$$

$$= \frac{5}{3} \quad //$$

Q) Give a definition of $\iint \vec{A}\cdot\hat{n}\,ds$ over a surface S in terms of limit of a sum:



Let us subdivide the area S into n elts of area Δs_p where $p = 1, 2, 3 \ldots n$. Let P_p be any point within the area Δs_p whose co-ordinates are (x_p, y_p, z_p).

Let $\vec{A}(x_p, y_p, z_p) = \vec{A}_p$

Let $\vec{A}$ be a vector funcⁿ whose value at the point P_p is $\vec{A}_p$ & let $\hat{n}$ be the +ve unit normal to Δs_p at the point P_p +ve pt P_p. Hence $\vec{A}_p \cdot \hat{n}_p$ is the normal component of A_p at P_p

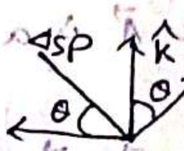
Then taking the limit of the sum $\sum_{P=1} \vec{A}_P \cdot \hat{n}_P \Delta S_P$ as $M \to \infty$ such that $\Delta S_P \to 0$. If the limit exists this limit is called the surface integral of normal component of $\vec{A}$ over S & is denoted by $\iint_S \vec{A} \cdot \hat{n}\, dS$

$$\therefore \iint_S \vec{A} \cdot \hat{n}\, dS = \lim_{\Delta S \to 0} \sum_{P=1}^{\infty} \vec{A}_P \cdot \hat{n}_P \Delta S_P$$

If the surface S has projection R on the x-y plane, so that $\iint_S \vec{A} \cdot \hat{n}\, dS = \iint_S \vec{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$

The projection of the area ΔS_P on the x-y plane

$$\Delta S_P \cos\theta = \Delta x \Delta y$$



$$\Delta S_P |\hat{n}_P \cdot \hat{k}| = \Delta x \Delta y$$

$$\Delta S_P = \frac{\Delta x \Delta y}{|\hat{n}_P \cdot \hat{k}|}$$

$$\iint_S \vec{A} \cdot \hat{n}\, dS = \iint_S \vec{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$\iint_S \vec{A} \cdot \hat{n}\, dS = \iint \lim_{\Delta S_P \to 0} \sum_{P=1}^{\infty} \vec{A}_P \cdot \hat{n}_P \frac{\Delta x \Delta y}{|\hat{n}_P \cdot \hat{k}|}$$

In the limit

$$\iint_S \vec{A} \cdot \hat{n}\, dS = \iint_S \vec{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

Integral Theorems :

① Gauss's theorem of divergence :-

Suppose, V is the vol^m bounded by a closed surface S & $\vec{A}$ is a vector funct^n of pos^n with continuous derivatives. Then,

$$\iiint \vec{\nabla}\cdot\vec{A}\, dV = \iint_S \vec{A}\cdot\hat{n}\, dS = \oiint_S \vec{A}\cdot d\vec{S}$$

where, $\hat{n}$ is the +ve outward drawn normal to S.

② Stoke's Theorem :-

Suppose, S is an open two sides surface bounded by a closed non intersecting curve C (a simple closed curve) & suppose $\vec{A}$ is a vector func^n of pos^n with continuous derivatives. Then,

$$\oint_C \vec{A}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \iint_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, dS$$

where, C is traversed in a +ve direc^n (anticlockwise)

③ Green's thorem in a plane :

Suppose, R is a closed region in the x-y plane bounded by a simple closed curve C & suppose, M & N are continuous func^{n}s of x & y having continuous derivatives in R. Then,

$$\oint_C M dx + N dy = \iint_R \left(-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right) dx\, dy$$

or

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

where, C is traversed in +ve direc^n (anticlockwise)

Green's theorem is a special case of Stoke's Theorem. Also, Gauss's Theorem is a generalisation of Green's theorem in the plane where the plane region R & its closed boundary curve C are replaced by a space region V & its closed boundary the surface S. For this reason, divergence theorem is often called Green's theorem in space. There are related integral theorems —

The following laws hold —

① $$\iiint_V \left[\phi \nabla^2 \psi + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\psi) \right] dV = \iint_S (\phi \vec{\nabla}\psi)\cdot d\vec{S}$$

This is Green's 1st identity.

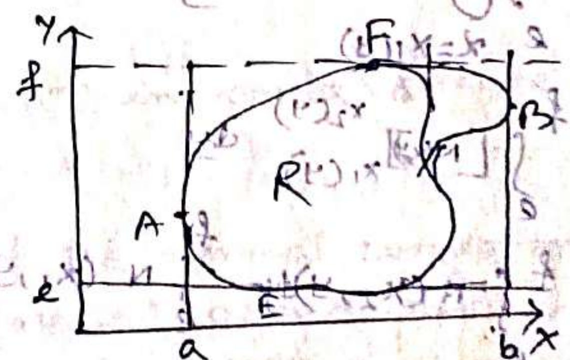
② $$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \iint_S (\phi \vec{\nabla}\psi - \psi \vec{\nabla}\phi)\cdot d\vec{S}$$

This is Green's 2nd identity.

③ $$\iiint_V \vec{\nabla}\times\vec{A}\, dV = \iint_S (\hat{n}\times\vec{A})\, dS = \iint_S d\vec{S}\times\vec{A}$$

④ $$\oint_C \phi\, d\vec{r} = \iint_S (\hat{n}\times\vec{\nabla}\phi)\, dS = \iint_S d\vec{S}\times\vec{\nabla}\phi$$

Since C is a closed curve, it has the property that any straight line parallel to the co-ordinate axis cuts C in at-most two points.



Let the eqns of the curve AEB & AFB be

$$y = Y_1(x) \;\&\; y = Y_2(x) \text{ respectively}$$

If R is a region bounded by C we have

$$\iint_R \frac{\partial M}{\partial y}\, dx\, dy = \int_{x=a}^{x=b} \left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\, dy \right] dx$$

$$= \int_a^b [M]_{Y_1(x)}^{Y_2(x)}\, dx$$

$$= \int_a^b [M(x, y_2) - M(x, y_1)]\, dx$$

$$= -\int_a^b M(x, y_1)\, dx - \int_b^a M(x, y_2)\, dx$$

$$= \oint M(x, y)\, dx$$

$$\therefore \; -\iint_R \frac{\partial M}{\partial y}\, dx\, dy = \oint M(x, y)\, dx \quad \text{(i)}$$

lly, let the eqn's of curves EAF & EBF be ~~x~~ $x = X_1(y)$ $x = X_2(y)$ respectively.

then

$$\iint_R \frac{\partial N}{\partial x}\, dx\, dy$$

$$= \int_e^f \left[\int_{x = X_1(y)}^{x = X_2(y)} \frac{\partial N}{\partial x}\, dx\right] dy$$

$$= \int_e^f [N(x, y)]_{X_1(y)}^{X_2(y)}\, dy$$

$$= \int_e^f N(X_2, y)\, dy - \int_e^f N(X_1, y)\, dy$$

$$= \int_e^f N(X_2, y)\, dy + \int_f^e N(X_1, y)\, dy$$

$$= \oint N(x, y)\, dy$$

Adding ① & ②
$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

Q. Express Green's Th^m in the plane in vector notation.

$\Rightarrow$ we have $M dx + N dy$

$= (M\hat{i} + N\hat{j}) \cdot (\hat{i}dx + \hat{j}dy)$

$= \vec{A} \cdot d\vec{r}$ where $\vec{A} = M\hat{i} + N\hat{j}$

Also, $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$

$= \hat{i}\left(-\frac{\partial N}{\partial z}\right) + \hat{j}\left(\frac{\partial M}{\partial z}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

Hence ~~But in~~ Green's th^m in the plane ~~is~~

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

can be written as

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_R (\vec{\nabla} \times \vec{A}) \cdot \hat{k}\, dx\,dy.$$

Q. Express the divergence th^m in words & write it in rectangular form.

$\Rightarrow$ The surface integral of the normal component of a vector $\vec{A}$ taken over a closed surface is equal to the integral of the divergence of $\vec{A}$ taken over the vol^m & closed by the surface.

Let $\vec{A} = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$

$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x}A_1 + \frac{\partial}{\partial y}A_2 + \frac{\partial A_3}{\partial z}$

$$\iiint_V \vec{\nabla} \cdot \vec{A}\, dv = \iint_S \vec{A} \cdot \hat{n}\, ds$$

(iii) $\vec{A} = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$

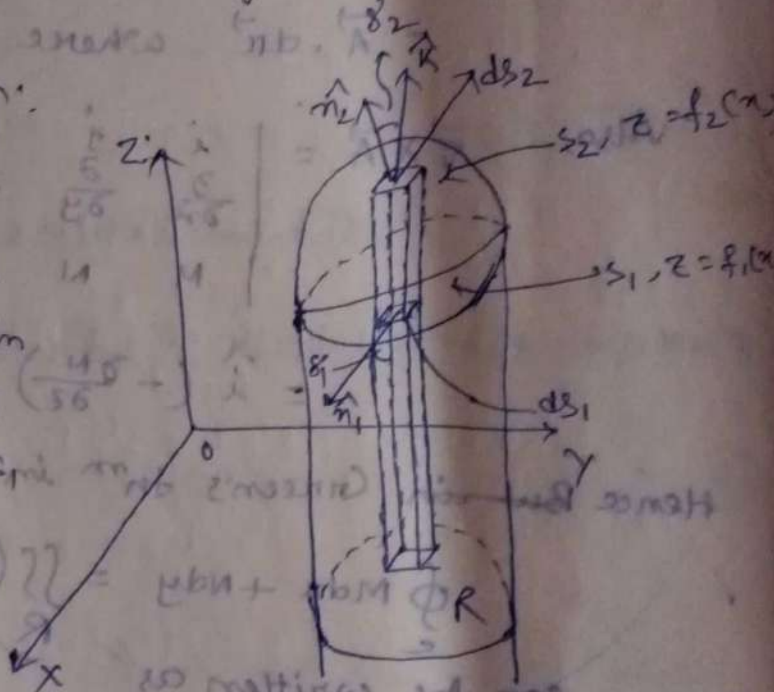
$\hat{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$, where α, β, γ are direc^n cosine of the unit normal.

$\therefore \vec{A}\cdot\hat{n}\, ds = A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma$

Gauss theorem,

$$\iint_S (A_1\cos\alpha + A_2\cos\beta + A_3\cos\gamma)\, ds = \iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dv$$

<u>Proof of the divergence theorem.</u>



Let S be a closed surface such that any line parallel to the co-ordinate axis cuts S in atmost 2 points. Let the eqn of the lower & the upper portion S_1 & S_2 be $z = f_1(x,y)$ & $z = f_2(x,y)$. Let R be the projection of the surface on the x-y plane.

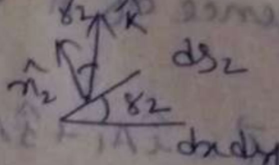
then, $\iiint_V \frac{\partial A_3}{\partial z}\, dx\, dy\, dz$

$$= \iint_V \left[A_3(x,y,z)\right]_{z=f_1(x,y)}^{z=f_2(x,y)} dx\, dy$$

$$= \iint \left[A_3(x,y,f_2) - A_3(x,y,f_1)\right] dx\, dy \quad \text{—— (i)}$$

Now $\hat{n}_2\cdot\hat{k}\, ds_2 = \cos\gamma_2\, ds$

$\therefore ds_2\cos\gamma_2 = dx\, dy$ —— (ii), for the upper portion.

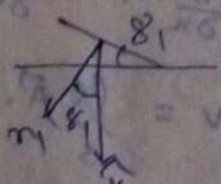


$ds_2\cos\gamma_2 = dx\, dy$

lly for the lower portion S_1,

$dx\, dy = -\hat{n}_1\cdot\hat{k}\, ds_1$

$= -\cos\gamma_1\, ds_1 \longrightarrow$ (iii)



Using (ii) & (iii) in (i),

$$\iiint_V \frac{\partial A_3}{\partial z}\,dx\,dy\,dz$$

$$= \iint_{S_2} A_3(x,y,f_2)\,\hat{n}_2\cdot\hat{k}\,ds_2 + \iint_{S_1} A_3(x,y,f_1)\,\hat{n}_1\cdot\hat{k}\,ds_1$$

$$= \iint_{S} A_3\,\hat{n}\cdot\hat{k}\,ds$$

$$= \iint_{S} A_3\,\hat{k}\cdot\hat{n}\,ds$$

$$\therefore \iiint_V \frac{\partial A_3}{\partial z}\,dv = \iint_S A_3\,\hat{k}\cdot\hat{n}\,ds \quad -(iv)$$

lly by projecting the surface S on the other co-ordinate plane

$$\iiint_V \frac{\partial A_1}{\partial x}\,dv = \iint_S A_1\,\hat{i}\cdot\hat{n}\,ds \quad -(v)$$

$$\iiint_V \frac{\partial A_2}{\partial y}\,dv = \iint_S A_2\,\hat{j}\cdot\hat{n}\,ds \quad -(vi)$$

Adding (iv), (v) & (vi) $\Rightarrow$ $\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)dv$

$$= \iint_S \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)\cdot\hat{n}\,ds$$

$$\Rightarrow \iiint_V \vec{\nabla}\cdot\vec{A}\,dv = \iint \vec{A}\cdot\hat{n}\,ds \,//$$

① Evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$ where S is a closed surface.

soln: Accn to Gauss theorem,

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{r}\,dv$$

$$= \iiint 3\,dv$$

$$= 3V \,//$$

Now $\vec{\nabla}\cdot\vec{r}$, $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 3$$

2) Prove $\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dv = \iint_S (\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi) \cdot d\vec{s}$

$\Rightarrow$ Let $\vec{A} = \phi \nabla \psi$

$\therefore \iiint_V \vec{\nabla} \cdot \vec{A}\, dv = \iiint_V \vec{\nabla} \cdot (\phi \vec{\nabla} \psi)\, dv = \iint_S (\phi \nabla \psi) \cdot d\vec{s}$

from Gauss' theorem.

Now $\iiint_V \vec{\nabla} \cdot (\phi \vec{\nabla} \psi)\, dv = \iiint (\vec{\nabla} \phi \cdot \vec{\nabla} \psi + \phi \nabla^2 \psi)\, dv$ —(1)

lly, $\iiint_V \vec{\nabla} \cdot (\psi \nabla \phi)\, dv = \iiint_V (\vec{\nabla} \psi \cdot \vec{\nabla} \phi + \psi \nabla^2 \phi)\, dv$ —(2)

Also $\iiint_V \vec{\nabla} \cdot (\phi \vec{\nabla} \psi)\, dv = \iint_S \phi \vec{\nabla} \psi \cdot d\vec{s}$ (3) & $\iiint_V \vec{\nabla} \cdot (\psi \vec{\nabla} \phi)\, dv = \iint_S \psi \vec{\nabla} \phi \cdot d\vec{s}$ (4)

Now (1) & (4) $\Rightarrow$ $\iint_S \phi \vec{\nabla} \psi \cdot d\vec{s} = \iiint_V (\vec{\nabla} \phi \cdot \vec{\nabla} \psi + \phi \nabla^2 \psi)\, dv$ —(5)

& $\iint_S \psi \vec{\nabla} \phi \cdot d\vec{s} = \iiint_V (\vec{\nabla} \psi \cdot \vec{\nabla} \phi + \psi \nabla^2 \phi)\, dv$ —(6)

(5) − (6) $\Rightarrow$ $\iiint (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dv = \iint_S (\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi)\, ds$ —(7)

eq^n (5) is Green's 1st identity & (7) is Green's 2nd identity

(3) Prove $\iiint_V \vec{\nabla} \phi\, dv = \iint_S \phi \hat{n}\, ds$

$\Rightarrow$ Let $\vec{A} = \phi \vec{c}$ where $\vec{c}$ is a constant vector from Gauss theorem

we have $\iiint_V \vec{\nabla} \cdot \vec{A}\, dv = \iint_S \vec{A} \cdot \hat{n}\, ds$ —(1)

$\iiint_V \vec{\nabla} \cdot \phi \vec{c}\, dv = \iiint_V (\phi \vec{\nabla} \cdot \vec{c} + \vec{\nabla} \phi \cdot c)\, dv$

But $\vec{\nabla} \cdot \vec{c} = 0$ because $\vec{c}$ is a const. vector

$= \vec{c} \cdot \iiint_V \vec{\nabla} \phi\, dv$ —(2)

L.H.S. of (1) $= \iiint_V \vec{\nabla} \phi \cdot \vec{c}\, dv$

R.H.S. of (1) $\Rightarrow \iint_S \vec{A} \cdot \hat{n}\, ds = \iint_S \phi \vec{c} \cdot \hat{n}\, ds = \vec{c} \cdot \iint_S \phi \hat{n}\, ds$ —(3)

Hence from (2) & (3) $\vec{c} \cdot \iiint_V \vec{\nabla} \phi\, dv = \vec{c} \cdot \iint_S \phi \hat{n}\, ds$

$\Rightarrow \vec{c} \cdot \left[\iiint_V \vec{\nabla} \phi\, dv - \iint_S \phi \hat{n}\, ds \right] = 0$

but $\vec{c}$ is an arbitary vector and $\vec{c} \neq 0$

$$\therefore \iiint_V \vec{\nabla}\phi\, dv - \iint_S \phi\hat{n}\, ds = 0$$

$$\Rightarrow \iiint_V \vec{\nabla}\phi\, dv = \iint_S \phi\hat{n}\, ds$$

(4) Prove $\iiint_V \vec{\nabla}\times\vec{B}\, dV = \iint_S \hat{n}\times\vec{B}\, ds$

$\Rightarrow$ Let $\vec{A} = \vec{B}\times\vec{C}$ where $\vec{C}$ is a const. vector.

Then from Gauss' theorem,

$$\iiint_V \vec{\nabla}\cdot\vec{A}\, dv = \iint_S \vec{A}\cdot\hat{n}\, ds$$

$$\iiint_V \vec{\nabla}\cdot(\vec{B}\times\vec{C})\, dv = \iint_S \vec{B}\times\vec{C}\cdot\hat{n}\, ds \quad -(1)$$

Now $\vec{\nabla}\cdot\vec{B}\times\vec{C} = \vec{C}\cdot\vec{\nabla}\times\vec{B} - B\cdot\vec{\nabla}\times\vec{C}$

but $\vec{\nabla}\times\vec{C} = 0$

$\therefore \vec{\nabla}\cdot\vec{B}\times\vec{C} = \vec{C}\cdot\vec{\nabla}\times\vec{B} \quad -(2)$

Now, $\vec{B}\times\vec{C}\cdot\hat{n}$

$= \hat{n}\cdot\vec{B}\times\vec{C}$

$= \vec{C}\cdot\hat{n}\times\vec{B} \quad -(3)$

using (2) & (3) in (1) we obtain,

$$\iiint_V \vec{C}\cdot\vec{\nabla}\times\vec{B}\, dV = \iint_S \vec{C}\cdot\hat{n}\times\vec{B}\, ds$$

$$\Rightarrow \vec{C}\cdot\left[\iiint_V \vec{\nabla}\times\vec{B}\, dv - \iint_S \hat{n}\times\vec{B}\, ds\right] = 0$$

But $C \neq 0$ because it is an ~~abt~~ arbitary vector

$$\therefore \iiint \vec{\nabla}\times\vec{B}\, dv - \iint_S \hat{n}\times\vec{B}\, ds = 0$$

$$\Rightarrow \iiint_V \vec{\nabla}\times\vec{B}\, dv = \iint_S \hat{n}\times\vec{B}\, ds$$

① Suppose S is any closed surface enclosing a volume V & $\vec{A} = ax\hat{i} + by\hat{j} + cz\hat{k}$ Prove that $\iint_S \vec{A}\cdot\hat{n}\,ds = (a+b+c)V$

Soln: From Gauss's theorem of divergence -

$$\iiint_V \vec{\nabla}\cdot\vec{A}\,dv = \iint_S \vec{A}\cdot\hat{n}\,ds$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}ax + \frac{\partial}{\partial y}by + \frac{\partial}{\partial z}cz$$

$$= a+b+c$$

$$\therefore \iiint_V \vec{\nabla}\cdot\vec{A}\,dv$$

$$= \iiint_V (a+b+c)\,dv$$

$$= (a+b+c)\iiint_V dv$$

$$= (a+b+c)V$$

② Suppose $\vec{H} = \text{curl}\,\vec{A}$ prove that $\iint_S \vec{H}\cdot\hat{n}\,ds = 0$ for any closed surface S.

$\Rightarrow$ From Gauss' theorem $\iint_S \vec{H}\cdot\vec{n}\,ds = \iiint_V \vec{\nabla}\cdot\vec{H}\,dv$

But $\text{div}(\text{curl}\,\vec{H}) = 0$

$$\therefore \iint_S \vec{H}\cdot\hat{n}\,ds = 0$$

③ Suppose $\hat{n}$ is the outward drawn unit normal to any closed surface of area S. Show that, $\iiint_V \text{div}\,\hat{n}\,dV = S$

$\Rightarrow$ From Gauss' theorem,

$$\iiint_V \vec{\nabla}\cdot\hat{n}\,dv = \iint_S \hat{n}\cdot\hat{n}\,ds = \iint_S ds = S$$

④ Prove $\iiint_V \frac{dv}{r^2} = \iint_S \frac{\vec{r}\cdot\hat{n}}{r^2}\, ds$

$\Rightarrow$ Let $\vec{A} = \frac{\vec{r}}{r^2}$

then $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^2}\right)$

$= \vec{\nabla}\left(\frac{1}{r^2}\right)\cdot\vec{r} + \frac{1}{r^2}(\vec{\nabla}\cdot\vec{r})$

$= -2r^{-4}\,\vec{r}\cdot\vec{r} + \frac{3}{r^2}$

$= \frac{3}{r^2} - \frac{2}{r^2}$

$= \frac{1}{r^2}$

from Gauss' Theorem,

$$\oint\!\!\oint_S \frac{\vec{r}\cdot\hat{n}}{r^2}\, ds = \iiint_V \vec{\nabla}\cdot\frac{\vec{r}}{r^2}\, dv$$

$$= \iiint_V \frac{1}{r^2}\, dv \quad //$$

⑤ Prove $\iint_S r^5\hat{n}\, ds = \iiint_V 5r^3\vec{r}\, dv$

$\Rightarrow$ Let $\vec{A} = r^5\vec{c}$ where $\vec{c}$ is a const. vector.

then $\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_S r^5\vec{c}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot r^5\vec{c}\, dv$

$\Rightarrow \vec{c}\cdot\iint_S r^5\hat{n}\, ds = \vec{c}\cdot\iiint_V 5r^3\vec{r}\, dv$

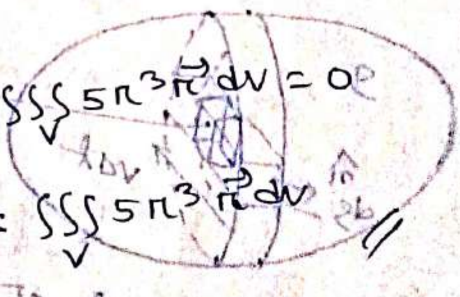
$\Rightarrow \vec{c}\cdot\left[\iint_S r^5\hat{n}\, ds - \iiint_V 5r^3\vec{r}\, dv\right] = 0$

$\vec{\nabla}\cdot r^5\vec{c} = \nabla r^5\cdot\vec{c} + r^5\,\vec{\nabla}\cdot\vec{c} = \nabla r^5\cdot\vec{c} + 0 = 5r^3\vec{r}\cdot\vec{c}$

$\because \vec{c} \neq 0$

$\therefore \iint_S r^5\hat{n}\, ds - \iiint_V 5r^3\vec{r}\, dv = 0$

$\Rightarrow \iint_S r^5\hat{n}\, ds = \iiint_V 5r^3\vec{r}\, dv \quad //$



⑥ Prove $\iint_S \hat{n}\, ds = 0$ for any closed surface S.

$\Rightarrow$ From Gauss' theorem $\iint_S \vec{A}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{A}\, dv$

If $\vec{A}$ is a const. vector.

$$\therefore \quad \vec{A}\cdot\iint_S \hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{A}\, dv = 0$$

$$\vec{A}\cdot\iint \hat{n}\, ds = 0$$

But $\vec{A}$ is an arbitary constant vector and $\vec{A}\neq 0$

$$\therefore \iint_S \hat{n}\, ds = 0 \text{ for any closed surface S.}$$

⑦ Prove $\iiint_V \vec{\nabla}\phi\cdot\vec{A}\, dv = \iint_S \phi\vec{A}\cdot\hat{n}\, ds - \iiint_V \phi\vec{\nabla}\cdot\vec{A}\, dv$

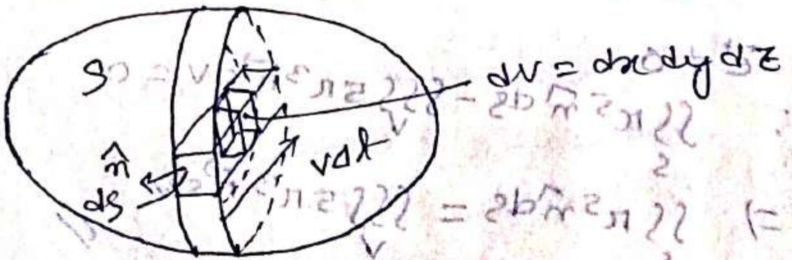
$\Rightarrow$ From Gauss' theorem:

$$\iiint_V \vec{\nabla}\cdot(\phi\vec{A})\, dv = \iint_S \phi\vec{A}\cdot\hat{n}\, ds$$

$$\text{or} \quad \iiint_V \phi\vec{\nabla}\cdot\vec{A}\, dv + \iiint_V \vec{\nabla}\phi\cdot\vec{A}\, dv = \iint_S \phi\vec{A}\cdot\hat{n}\, ds$$

$$\Rightarrow \iiint_V \vec{\nabla}\phi\cdot\vec{A}\, dv = \iint_S \phi\vec{A}\cdot\hat{n}\, ds - \iiint_V \phi\vec{\nabla}\cdot\vec{A}\, dv$$

① Demonstrate the divergence theorem physically.



Let $\vec{v}$ be the velocity at any point of a moving fluid. Then the volume of fluid contained in a cylinder of base ds and height $\vec{v}\Delta t$ = volume of fluid crossing ds in Time Δt secs.

$$= \vec{v}\Delta t\cdot\hat{n}\, ds$$
$$= \vec{v}\cdot\hat{n}\, ds\,\Delta t$$

∴ volume of fluid crossing the area ds per unit time -

$= \vec{v}.\hat{n} ds$

Total volume of fluid emerging from the close surface S per unit time $= \iint_S \vec{v}.\hat{n} ds$

But Also, the loss of fluid per unit volume per unit time in a small volume $dv = dx dy dz$ is given by $\vec{\nabla}.\vec{v}$ i.e $\vec{\nabla}.\vec{v} dv$ is the volume of fluid emerging per sec from a volm elt. dv. Then the total volm of fluid emerging per sec from all the volm elts in S $= \iiint_V \vec{\nabla}.\vec{v} dv$.

$$\therefore \iiint_V \vec{\nabla}.\vec{v} dv = \iint_S \vec{v}.\hat{n} ds$$

(2) Express Stroke's theorem in words and write in rectangular form.

⇒ The line integral of the trangential component of a vector $\vec{A}$ taken around a simple closed curve c is equal to the surface integral of the normal component of $\vec{\nabla}\times\vec{A}$ taken over any surface S having c as its boundary.

$$\vec{A} = \hat{i} A_1 + \hat{j} A_2 + \hat{k} A_3$$

$$\hat{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$$

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\vec{\nabla}\times\vec{A}).\hat{n} ds$$

$$\oint_C \vec{A}.d\vec{r} = \oint_C A_1 dx + A_2 dy + A_3 dz$$

$$\iint_S (\vec{\nabla}\times\vec{A}).\hat{n} ds = \iint_S \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} . (\hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma) ds$$

$$= \iint_S \left[\hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \hat{j}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right] \cdot (\hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma)\, ds$$

$$= \iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma\right] ds$$

$$= \oint_C A_1\,dx + A_2\,dy + A_3\,dz$$

$\therefore$ is Stokes' thm in rectangular form.

① Prove $\oint d\vec{r} \times \vec{B} = \iint_S (\hat{n} \times \vec{\nabla}) \times \vec{B}\, ds$

Pf: Let $\vec{A} = \vec{B} \times \vec{C}$

where $\vec{C}$ is a const. vector

According to Stokes' Thm $\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, ds$

$$\Rightarrow \oint_C \vec{B} \times \vec{C} \cdot d\vec{r} = \iint_S \vec{\nabla} \times (\vec{B} \times \vec{C}) \cdot \hat{n}\, ds$$

$$\Rightarrow \oint_C d\vec{r} \cdot (\vec{B} \times \vec{C}) = \iint_S$$

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla} \cdot \vec{A}) - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B})$$

$$\vec{\nabla} \times (\vec{B} \times \vec{C}) = (\vec{C} \cdot \vec{\nabla})\vec{B} - \vec{C}(\vec{\nabla} \cdot \vec{B}) - (\vec{B} \cdot \vec{\nabla})\vec{C} + \vec{B}(\vec{\nabla} \cdot \vec{C})$$

But $\vec{C}$ = const. $\therefore$ $\vec{\nabla} \cdot \vec{C} = 0$ and $(\vec{B} \cdot \vec{\nabla})\vec{C} = 0$

$$\therefore \oint_C d\vec{r} \cdot (\vec{B} \times \vec{C}) = \iint_S [(\vec{C} \cdot \vec{\nabla})\vec{B} - \vec{C}(\vec{\nabla} \cdot \vec{B})] \cdot \hat{n}\, ds$$

$$\Rightarrow \oint_C \vec{C} \cdot (d\vec{r} \times \vec{B}) = \iint_S [\vec{C} \cdot \vec{\nabla}(\vec{B} \cdot \hat{n}) - \vec{C} \cdot \hat{n}(\vec{\nabla} \cdot \vec{B})]\, ds$$

$$\Rightarrow \vec{C} \cdot \oint_C d\vec{r} \times \vec{B} = \vec{C} \cdot \iint_S [\vec{\nabla}(\vec{B} \cdot \hat{n}) - \hat{n}(\vec{\nabla} \cdot \vec{B})]\, ds$$

$$\Rightarrow \vec{C} \cdot \oint_C d\vec{r} \times \vec{B} = \vec{C} \cdot \iint_S (\hat{n} \times \vec{\nabla}) \times \vec{B}\, ds$$

$$\Rightarrow \vec{C} \cdot \left[\oint_C d\vec{r} \times \vec{B} - \iint_S (\hat{n} \times \vec{\nabla}) \times \vec{B}\, ds\right] = 0$$

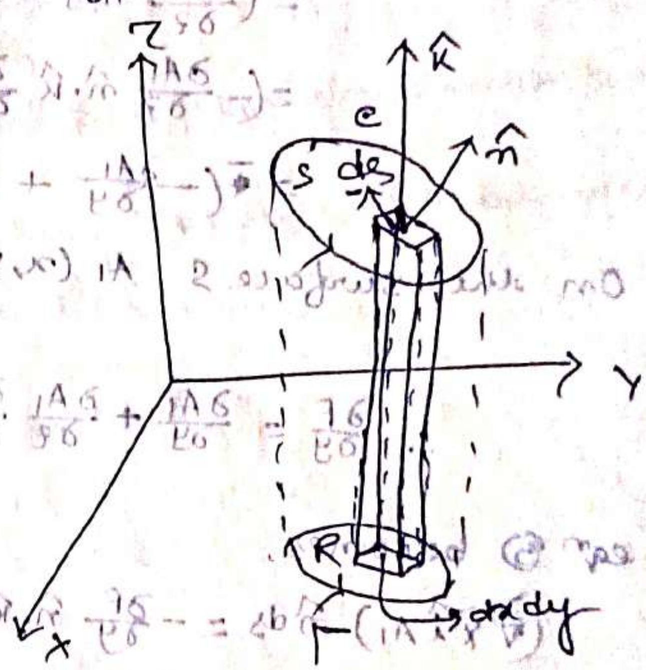
$\therefore \oint_C d\vec{R} \times \vec{B} = \iint_S (\hat{n} \times \vec{\nabla}) \times \vec{B}\, ds$

Pf. of Stoke's thm

Let S be a surface such that its projection on the x-y, y-z & z-x planes are regions bounded by simple closed curves. Let S be represented by the curves $z = f(x, y)$

$$y = g(z, x)$$
$$z = h(x, y)$$

where f, g, h are single valued continuous and differential funcⁿs. C is the boundary of surface S.



According to stoke's thm

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, ds$$

$$\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, ds$$

Now $\vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\hat{i} A_1 + \hat{j} A_2 + \hat{k} A_3)$

$\therefore \vec{\nabla} \times (\hat{i} A_1)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix}$$

$$= \hat{j}\left(\frac{\partial A_1}{\partial z}\right) - \hat{k}\frac{\partial A_1}{\partial y}$$

$\therefore (\vec{\nabla} \times \hat{i} A_1) \cdot \hat{n}\, ds$

$$= \left(\hat{j}\frac{\partial A_1}{\partial z} - \hat{k}\frac{\partial A_1}{\partial y}\right) \cdot \hat{n}\, ds$$

$$= \left(\frac{\partial A_1}{\partial z}\, \hat{n} \cdot \hat{j} - \frac{\partial A_1}{\partial y}\, \hat{n} \cdot \hat{k}\right) ds \longrightarrow (1)$$

If $z = f(x, y)$ is taken as a eqⁿ of S then the posⁿ vector to any point of S is $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z = \hat{i}x + \hat{j}y + \hat{k} f(x, y)$

Again $\frac{\partial \vec{r}}{\partial y} = \hat{j} + \hat{k}\frac{\partial f}{\partial y}$

But $\frac{\partial \vec{r}}{\partial y}$ is a vector tangent to S. ~~But $\frac{\partial \vec{r}}{}$~~. Hence it is perpendicular to $\hat{n}$.

$$\hat{n}\cdot\frac{\partial \vec{r}}{\partial y} = 0$$

$$\Rightarrow \hat{n}\cdot\left(\hat{j} + \hat{k}\frac{\partial f}{\partial y}\right) = 0$$

$$\therefore \hat{n}\cdot\hat{j} + \hat{n}\cdot\hat{k}\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \hat{n}\cdot\hat{j} = -\hat{n}\cdot\hat{k}\frac{\partial f}{\partial y} \quad —(2)$$

Using (2) in (1) $(\vec{\nabla}\times\hat{i}A_1)\cdot\hat{n}\,ds$

$$= \left(\frac{\partial A_1}{\partial z}\hat{n}\cdot\hat{j} - \frac{\partial A_1}{\partial y}\hat{n}\cdot\hat{k}\right)ds$$

$$= \left(-\frac{\partial A_1}{\partial z}\hat{n}\cdot\hat{k}\frac{\partial f}{\partial y} - \frac{\partial A_1}{\partial y}\hat{n}\cdot\hat{k}\right)ds$$

$$= -\left(\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\right)\hat{n}\cdot\hat{k}\,ds \quad —(3)$$

On the surface S $\quad A_1(x,y,z) = A_1(x,y,f(x,y))$

$$= F(x,y)$$

$$\therefore \frac{\partial F}{\partial y} = \frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}$$

$\therefore$ eqn (3) becomes,

$$(\vec{\nabla}\times\hat{i}A_1)\cdot\hat{n}\,ds = -\frac{\partial F}{\partial y}\hat{n}\cdot\hat{k}\,ds \quad —(4)$$

R is the projection of S in the xy plane. Therefore $\hat{n}\cdot\hat{k}\,ds = dx\,dy$ is the projection of area ds on the xy plane.

$$\therefore \iint_S (\vec{\nabla}\times\hat{i}A_1)\cdot\hat{n}\,ds = -\iint_R \frac{\partial F}{\partial y}\,dx\,dy \rightarrow (5)$$

~~where~~ By Green's thm for the plane,

$$\iint_R \frac{\partial F}{\partial y}\,dx\,dy = \oint_\Gamma F\,dx \quad \text{where } \Gamma \text{ is the boundary of } R$$

~~where~~ Since at each point x, y of Γ the value of F is the same as the value of A_1 at each point x, y, z at C, since dx is same for both curves. we must have

$$\oint_{\Gamma} F\,dx = \oint_{C} A_1\,dx$$

~~lly by~~ $$\iint (\vec{\nabla} \times \hat{i} A_1) \cdot \hat{n}\,ds = \oint_{C} A_1\,dx \quad -(6)$$

lly by projection on the co-ordinate plane

$$\iint_S (\vec{\nabla} \times \hat{j} A_2) \cdot \hat{n}\,ds = \oint_C A_2\,dy \quad -(7)$$

$$\iint_S (\vec{\nabla} \times \hat{k} A_2) \cdot \hat{n}\,ds = \oint A_3\,dz \quad -(8)$$

Now, (6) + (7) + (8) $\Rightarrow$ $$\iint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\,ds = \oint \vec{A} \cdot d\vec{r}$$

Q. A vector $\vec{B}$ is always normal to a given closed ~~curve~~ surface S so that,

$$\iiint_V \text{curl}\,\vec{B}\,dv = 0$$, where V is the volm bounded by S

$\Rightarrow$ Gauss thm,

$$\iiint_V \vec{\nabla} \cdot \vec{B}\,dv = \iint_S \vec{B} \cdot \hat{n}\,ds$$

$$\vec{\nabla} \cdot \vec{B} \times \vec{c} = \vec{c} \cdot \vec{\nabla} \times B - \vec{B} \cdot \vec{\nabla} \times \vec{c}_0$$

$$\iiint_V \vec{\nabla} \cdot \vec{B} \times \vec{c}\,dv = \iiint_V \vec{c} \cdot (\vec{\nabla} \times \vec{B})$$

$$\iiint \vec{\nabla} \times \vec{B}\,dv = \iint_S \hat{n} \times \vec{B}\,ds$$

But $\vec{R}$ is normal to the surface i.e $\vec{B} = \hat{n} B$

$$\therefore \hat{n} \times \vec{B} = \hat{n} \times \hat{n} B = \vec{0}$$

$$\therefore \iiint_V \text{curl}\,\vec{B}\,dv = 0$$