Power Series solution of differential equations

A power series is in general a series of the form :-

$$y = \sum_{m=0}^{\infty} c_m (x-a)^m$$

$$y = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

when $a = 0$

$$y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m + \cdots$$

c_0, c_1, c_2 are co-efficients of the series.

Let the coefficients along with x are real.

some special series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{\lfloor 2m} = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{\lfloor 2m+1} = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots$$

First assume a solution in the form of a power series, say,

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots = \sum_{m=0}^{\infty} c_m x^m \quad -① $$

$$y' = c_1 + 2c_2x + 3c_3x^2 + \cdots = \sum_{m=1}^{\infty} m c_m x^{m-1} \quad -② $$

$$y'' = 2c_2 + 3.2c_3x + \cdots = \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2} \quad -③ $$

Collect the terms containing same powers of x.

The result will be,

$$k_0 + k_1x + k_2x^2 + \cdots = 0 \quad -④$$

where the terms k_0, k_1, k_2 contain the unkown coefficiens c_0, c_1, c_2.

In (4) since $x \neq 0$

$$\therefore k_0 = k_1 = k_2 = 0$$

Thus $c_0, c_1, c_2 \cdots$ can be determined successively.

(V) Solve by power series method: $y' - y = 0$

$\Rightarrow$ Let $y = \sum_{m=0}^{\infty} c_0 x^m = c_0 + c_1x + c_2x^2 + \cdots$

$$y' = \sum_{m=1}^{\infty} m c_m x^{m-1} = c_1 + 2c_2x + 3c_3x^2 + \cdots$$

$$y'' = \sum_{m=2}^{\infty} m c_m (m-1) x^{m-2} = 2c_2 + 3.2c_3x + 4.3c_4x^2$$

substitute y' and y in eq ①

$$(c_1 + 2c_2x + 3c_3x^2 + \cdots) - (c_0 + c_1x + c_2x^2 + \cdots)$$

collect like powers of x.

$$(c_1 - c_0) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + \cdots = 0 \quad -②$$

equate the co-efficients of various powers of x into zero.

$\Rightarrow c_1 - c_0 = 0 \qquad 2c_2 - c_1 = 0, \qquad 3c_3 - c_2 = 0$

$\Rightarrow c_1 = c_0 \qquad c_2 = \frac{c_1}{2} = \frac{c_0}{2} \qquad c_3 = \frac{c_2}{3} = \frac{c_0}{3.2} = \frac{c_0}{\lfloor 3}$

$\therefore$ The assumed solution i.e

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 \cdots$$

becomes $= c_0 + c_0x + \frac{c_0}{2!}x^2 + \frac{c_0}{3!}x^3 + \cdots$

$= c_0\left(1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots\right)$

$= c_0 e^x$

(2) $y''+y=0$

Let the solution be,

$$y = c_0+c_1x+c_2x^2+\cdots$$

$$y' = c_1+2c_2x+3c_3x^2+\cdots$$

$$y'' = 2c_2+3.2\,c_3x+4.3.2.c_4x^2+\cdots$$

$$y''+y=0$$

$$\Rightarrow (2c_2+3.2.c_3x+4.3.2.c_4x^2+\cdots)+(c_0+c_1x+c_2x^2+\cdots)=0$$

$$\Rightarrow (2c_2+c_0)+(3.2\,c_3+c_1)x+(4.3.2\,c_4+c_2)x^2+\cdots=0$$

equating coefficients to zero.

$2c_2+c_0=0$ $\qquad$ $3.2.c_3+c_1=0$

$\Rightarrow 2c_2=-c_0$ $\qquad$ $\Rightarrow -c_1=3.2.c_3$

$\Rightarrow c_2=-\frac{c_0}{\lfloor 2}$ $\qquad$ $\Rightarrow c_3=-\frac{c_1}{\lfloor 3}$

$4.3.2\,c_4+c_2=0$

$$\Rightarrow c_4 = -\frac{c_2}{4.3}$$

$$\Rightarrow c_4 = -\frac{c_0}{4.3.2} = -\frac{c_0}{\lfloor 4}$$

c_0 and c_1 can be taken to be arbitrary

substituting : $y=c_0+c_1x+c_2x^2+c_3x^3+\cdots$

$$\Rightarrow y = c_0 + c_1x+\left(-\frac{c_0}{\lfloor 2}\right)x^2+\left(-\frac{c_1}{\lfloor 3}\right)x^3+\cdots$$

$$= c_0\left(1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\cdots\right)+c_1\left(x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}\right.$$

$$= c_0\cos x + c_1\sin x.$$

8) Solve $y' = 3y$ — ①

$\Rightarrow$ Let $y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

$y' = \sum_{m=1}^{\infty} m c_m x^{m-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$

Substituting y' and y in eq^n ①

$(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots) = 3(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$

$\Rightarrow (c_1 - 3c_0) + (2c_2 - 3c_1)x + (3c_3 - 3c_2)x^2 + (4c_4 - 3c_3)x^3 + \cdots = 0$

equating co-efficients to zero,

$c_1 - 3c_0 = 0$

$\Rightarrow c_1 = 3c_0$

$2c_2 - 3c_1 = 0$

$\Rightarrow 2c_2 = 3c_1$

$\Rightarrow c_2 = \frac{3c_1}{2} = \frac{3.3c_0}{2} = \frac{9c_0}{2}$

$3c_3 - 3c_2 = 0$

$\Rightarrow 3c_3 = 3c_2$

$\Rightarrow c_3 = c_2 = \frac{9c_0}{2}$

$4c_4 - 3c_3 = 0$

$\Rightarrow c_4 = \frac{3}{4}c_3 = \frac{27c_0}{8}$

$\therefore$ solution

$y = c_0 + 3c_0 x + \frac{9c_0}{2}x^2 + \frac{9c_0}{2}x^3 + \frac{27}{8}c_0 x^4 + \cdots$

$= c_0\left(1 + 3x + \frac{(3x)^2}{\lfloor 2} + \frac{(3x)^3}{\lfloor 3} + \frac{(3x)^4}{\lfloor 4} + \cdots\right)$

$= c_0 e^{3x}$

9) Solve $y' = ky$

$y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

$y' = \sum_{m=1}^{\infty} m c_m x^{m-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$

$y' = ky$

$\Rightarrow c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots = k(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$

$\Rightarrow (c_1 - kc_0) + (2c_2 - kc_1)x + (3c_3 - kc_2)x^2 + (4c_4 - kc_3)x^3 + \cdots$
$= 0$

equating the coefficients to zero.

$c_1 = kc_0$ $\quad$ $2c_2 - kc_1 = 0$ $\quad$ $3c_3 - kc_2 = 0$

$\Rightarrow c_2 = \frac{k}{2}c_1 = \frac{k^2}{2}c_0$ $\quad$ $\Rightarrow c_3 = \frac{k}{3}c_2$

$= \frac{k^3}{6}c_0$

$4c_4 - kc_3 = 0$

$\Rightarrow c_4 = \frac{k}{4}c_3$

$= \frac{k^4}{24}c_0$

$\therefore$ Soln. is

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

~~$= c_0 + \frac{x^2}{2}c_0 x + \frac{k^3}{6}c_0 x^2 + \frac{k^4}{24}c_0 x^3 + \cdots$~~

~~$= c_0 \left(1 + \frac{k^2}{6}x + \frac{k^3}{6}x^2 + \frac{k^4}{24}x^3 + \cdots\right)$~~

$= c_0 + kc_0 x + \frac{k^2}{2}c_0 x^2 + \frac{k^3}{6}c_0 x^3 + \cdots$

$= c_0 \left(1 + kx + \frac{(kx)^2}{\lfloor 2} + \frac{(kx)^3}{\lfloor 3} + \cdots\right)$

$= c_0 e^{kx}$

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(3) $y'' = y$

Let $y = \sum_{m=0}^{\alpha} c_m x^m = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$

$\therefore y' = \sum_{m=1}^{\alpha} m c_m x^{m-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$

$y'' = \sum_{m=2}^{\alpha} m(m-1) c_m x^{m-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$

$y'' = y$

$\Rightarrow 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$

$\Rightarrow (2c_2 - c_0) + (6c_3 - c_1)x + (12c_4 - c_2)x^2 + \cdots = 0$

equating co-efficients to zero

$2c_2 - c_0 = 0$ $\quad$ $6c_3 - c_1 = 0$ $\quad$ $12c_4 - c_2 = 0$

$\Rightarrow 2c_2 = c_0$ $\quad$ $\Rightarrow 6c_3 = c_1$ $\quad$ $\Rightarrow c_4 = \frac{c_2}{12} = \frac{c_0}{24}$

$\Rightarrow c_2 = \frac{c_0}{2}$ $\quad$ $\Rightarrow c_3 = \frac{c_1}{6}$

sol^n

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$\Rightarrow y = c_0 + c_1 x + \frac{c_0}{2} x^2 + \frac{c_1}{6} x^3 + \cdots$$

$$\Rightarrow y = c_0\left(1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots\right) + c_1\left(x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots\right)$$

(4) Solve $y'' = 4y$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots) = 4(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)$$

$$\Rightarrow (2c_2 - 4c_0) + (6c_3 - 4c_1)x + (12c_4 - 4c_2)x^2 + (20c_5 - 4c_3)x^3 + \cdots = 0$$

equating co-efficients to zero -

$2c_2 - 4c_0 = 0 \qquad 6c_3 - 4c_1 = 0 \qquad 12c_4 - 4c_2 = 0$

$\Rightarrow c_2 - 2c_0 = 0 \qquad \Rightarrow c_3 = \frac{2}{3} c_1 \qquad \Rightarrow c_4 = \frac{1}{3} c_2$

$\Rightarrow c_2 = 2c_0 \qquad\qquad\qquad\qquad = \frac{2}{3} c_0$

$20c_5 - 4c_3 = 0$

$\Rightarrow c_5 = \frac{c_3}{5}$

$= \frac{2}{15} c_1$

$\therefore$ sol^n:

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$= c_0 + c_1 x + 2c_0 x^2 + \frac{2}{3} c_1 x^3 + \frac{2}{3} c_0 x^4 + \cdots$$

$$= \left(c_0 + 2c_0 x^2 + \frac{2}{3} c_0 x^4 + \cdots\right) + \left(c_1 x + \frac{2}{3} c_1 x^3 + \cdots\right)$$

$$= c_0\left(1 + 2x^2 + \frac{2}{3} x^4 + \cdots\right) + c_1\left(x + \frac{2}{3} x^3 + \frac{2}{15} x^5 + \cdots\right)$$

Legendre's differential equation is given by,

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

n is a real number. Any solution of (1) is called Legendre's funcn.

Dividing throughout by $(1-x^2)$ we see that (1) reduces to a form where the co-efficients are analytic at $x=0$, hence power series method may be applied.

Let the solution be,

$$y = \sum_{m=0}^{\alpha} c_m x^m \quad \text{--- (2)}$$

$$y' = \sum_{m=1}^{\alpha} m c_m x^{m-1}$$

$$= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$y'' = \sum_{m=2}^{\alpha} m(m-1) c_m x^{m-2}$$

$$= 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots$$

$$(1) \Rightarrow (1-x^2)\sum_{m=2}^{\alpha} m(m-1) c_m x^{m-2} - 2x \sum_{m=1}^{\alpha} m c_m x^{m-1} + n(n+1)\sum_{m=0}^{\alpha} c_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\alpha} m(m-1) c_m x^{m-2} - \sum_{m=2}^{\alpha} m(m-1) c_m x^m - 2x\sum_{m=1}^{\alpha} m c_m x^{m-1} + n(n+1)\sum_{m=0}^{\alpha} c_m x^m = 0$$

put $n(n+1) = k$

Breaking up,

$$\Rightarrow 2.1\,c_2 + 3.2\,c_3 x + 4.3\,c_4 x^2 + \cdots + (s+2)(s+1)c_{s+2}x^s + \cdots$$

$$\cancel{c_2 + c_2 x^2 \cdots s(s-1)c}$$

$$\cancel{2c_1 x} \quad -2.1.\,c_2 x^2 \cdots\cdots - s(s-1)c_s x^s - \cdots$$

$$-2c_1 x \quad -2.2\,c_2 x^2 \cdots\cdots -2s c_s x^s - \cdots$$

$$kc_0 + kc_1 x + kc_2 x^2 \cdots\cdots k c_s x^s = 0$$

The last eqn is an identity, hence the co-efficients of each power of x must be zero.

$\Rightarrow \quad 2c_2 + kc_0 = 0 \quad , \quad 6c_3 - 2c_1 + kc_1 = 0$

$6c_3 + [-2 + n(n+1)]\, c_1 = 0$

in general, for $s = 2, 3 \ldots$

$$(s+2)(s+1)\, c_{s+2} + [-s(s-1) - 2s + n(n+1)]\, c_s = 0 \quad \text{--- (2)}$$

$-s(s-1) - 2s + n(n+1)$

$= -s^2 + s - 2s + n^2 + n$

$= n^2 - s^2 + n - s$

$= (n-s)(n+s+1)$

$\therefore$ (2) $\Rightarrow (s+2)(s+1)\, c_{s+2} + (n-s)(n+s+1)\, c_s = 0$

$$\Rightarrow \quad c_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}\, c_s \quad \text{--- (3)} \qquad s = 0, 1, 2 \ldots$$

eqn (3) is called recursion formula.

$$c_2 = \frac{-n(n+1)}{2!}\, c_0$$

$$c_3 = \frac{-(n-1)(n+2)}{3!}\, c_1$$

$$c_4 = \frac{-(n-2)(n+3)}{4 \cdot 3}\, c_2 = \frac{+(n-2)n(n+1)(n+3)}{4 \cdot 3 \cdot 2 \cdot 1}\, c_0$$

$$c_5 = \frac{-(n-3)(n+4)}{5 \cdot 4}\, c_3$$

$$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}\, c_1$$

Using in $y = \sum_{m=0}^{\infty} c_m x^m$

$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$

$$= c_0 + c_1 x - \frac{n(n+1)}{\lfloor 2}\, c_0 x^2 - \frac{(n+2)(n-1)}{\lfloor 3}\, c_1 x^3 + \frac{(n-2)n(n+1)(n+3)}{\lfloor 4}\, c_0 x^4 + \frac{(n-3)(n-1)(n+2)(n+4)}{\lfloor 5}\, c_1 x^5$$

$$y(x) = c_0 \left\{ 1 - \frac{n(n+1)}{\lfloor 2} x^2 + \frac{(n-2)n(n+1)(n+3)}{\lfloor 4} x^4 - \ldots \right\}$$

$$+ c_1 \left\{ x - \frac{(n+2)(n-1)}{\lfloor 3} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{\lfloor 5} x^5 - \ldots \right\}$$

Put $y_1(x) = \left\{1 - \frac{n(n+1)}{\lfloor 2}x^2 + \frac{(n-2)n(n+1)(n+3)}{\lfloor 4}x^4 - \cdots\right\}$

$y_2(x) = \left\{x - \frac{(n-1)(n+2)}{\lfloor 3}x^3 + \frac{(n-3)(n-1)(n+4)}{\lfloor 5}x^5 \cdots\right\}$

Then,

$$y(x) = c_0 y_1(x) + c_1 y_2(x) \quad -(4)$$

Legendre Polynomials

writing the recursion formula (4) in the following form,

$$c_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} c_{s+2} \quad \{s \le n-2\}$$

Then, the co-efficients can be expressed in terms of the co-efficients c_n of the highest power of the x in the polynomial.

c_n was chosen to be,

$$c_n = \frac{\lfloor 2n}{2^n(\lfloor n)^2}, \text{ so that } n=0 \text{ gives } c_n = 1.$$

$$= \frac{1.3.5\cdots(2n-1)}{\lfloor n}$$

$\lfloor 2n = 1.3.5\ldots(2n-1).2^n \lfloor n$

$$c_{n-2} = \frac{-n(n-1)}{2(2n-1)} c_n = \frac{-n(n-1)}{2(2n-1)} \frac{\lfloor 2n}{2^n(\lfloor n)^2}$$

$$= \frac{-\cancel{n}(\cancel{n-1})\,\cancel{2n}\,(\cancel{2n-1})\lfloor 2n-2}{\cancel{2}(\cancel{2n-1})\,2^n\,\cancel{n}\lfloor n-1\;\;\cancel{n}(\cancel{n-1})\lfloor n-2}$$

$$c_{n-2} = \frac{\lfloor 2n-2}{2^n \lfloor n-1 \lfloor n-2}$$

$$c_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} c_{n-2}$$

$$= \frac{(n-2)(n-3)}{4(2n-3)} \times \frac{\lfloor 2n-2}{2^n \lfloor n-1 \lfloor n-2}$$

$$= \cancel{\frac{(n-2)(n-3)}{4(2n-3)}} \times \frac{(2n-2)(\cancel{2n-3})\lfloor 2n-4}{2^n(\cancel{n-1})(\cancel{n-2})(\cancel{n-3})\lfloor n-4\;\cancel{\lfloor n-2}}$$

$$= \frac{(\cancel{n-2})(\cancel{n-3})(2n-2)(2n-3)\lfloor 2n-4}{4(2n-3)\,2^n(n-1)\lfloor n-2\,(\cancel{n-2})(\cancel{n-3})\lfloor n-4}$$

$$= \frac{\cancel{2}(\cancel{n-1})(\cancel{2n-3})\lfloor 2n-4}{2\cdot 4(\cancel{2n-3})\,2^n(\cancel{n-1})\lfloor n-2\,\lfloor n-4}$$

$$= \frac{\lfloor 2n-4}{\lfloor 2\; 2^n \lfloor n-2 \lfloor n-4}$$

In general,

$$C_{n-2m} = \frac{(-1)^m \lfloor 2n-2m}{2^n \lfloor m \lfloor n-m \lfloor n-2m}$$

$P_n(x)$ is the co-efficient of h^n in the expansion $(1-2xh+h^2)^{-1/2}$

we have. $(1-2xh+h^2)^{-1/2} = [1-h(2x-h)]^{-1/2}$

$$= 1+\frac{1}{2}h(2x-h)+\frac{1}{2}\cdot\frac{3}{4}\cdot h^2(2x-h)^2+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}h^3(2x-h)^3 + \cdots + \frac{1.3.5\ldots(2n-1)(2x-h)^n h^n}{2.4.6\ldots 2n}$$

Next, we find the co-efficient of h^n in the expansion.

Consider $\frac{1.3.5\ldots(2n-1)h^n(2x-h)^n}{2.4.6\ldots 2n}$

$$= \frac{1.3.5\ldots(2n-1)h^n}{2.4.6\ldots 2n}\left\{(2x)^n - {}^nC_1(2x)^{n-1}h + {}^nC_2(2x)^{n-2}h^2\ldots\right\}$$

coefficient of $h^n = \frac{1.3.5\ldots(2n-1)(2x)^n}{2.4.6\ldots 2n}$

consider $\frac{1.3.5\ldots(2n-3)h^{n-1}(2x-h)^{n-1}}{2.4.6\ldots(2n-2)}$

$$= \frac{1.3.5(2n-3)h^{n-1}}{2.4.6\ldots(2n-2)}\left\{(2x)^{n-1} - {}^{n-1}C_1(2x)^{n-2}h + {}^{n-1}C_2(2x)^{n-3}h^2 \ldots\right\}$$

coefficient of $h^n = -\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}{}^{n-1}C_1(2x)^{n-2}$

$$= -\frac{1.3.5\ldots(2n-3)(2n-1)}{2.4.6\ldots(2n-2)(2n-1)}\frac{(n-1)!}{(n-2)!}2^{n-2}x^{n-2}$$

$$= -\frac{1.3.5\ldots(2n-1)\,n(n-1)(n-2)!}{1.2.3\ldots(n-1)2^{n-1}\,n(2n-1)(n-2)!}2^{n-2}x^{n-2}$$

$$= -\frac{1.3.5\ldots(2n-1)\,n(n-1)x^{n-2}\,2^{n-2}}{n!\,2^{n-2}\,2(2n-1)}$$

$$= -\frac{1.3.5\ldots(2n-1)\,n(n-1)\,x^{n-2}}{n!\,2(2n-1)}$$

Consider, $\frac{1.3.5 \cdots (2n-5)}{2.4.6 \cdots (2n-4)} h^{n-2} (2x-h)^{n-2}\, {}^{n-2}C_2$

$$= \frac{1.3.5 \cdots (2n-5)(2n-3)(2n-1)\, 2^{n-4} x^{n-4}}{(2.2 \cdots n-2 \text{ terms})\{1.2.3 \cdots (n-2)\}(2n-3)(2n-1)}$$

$$= \frac{1.3.5 \cdots (2n-1)(n-1)n\, 2^{n-4} x^{n-4} (n-2)!}{2^{n-2}\, 1.2.3 \cdots (n-2)(n-1)n\, (2n-3)(2n-1)\, 2!\,(n-4)!}$$

$$= \frac{1.3.5 \cdots (2n-1)(n-1)n}{n!\, 2^2\, 2!\, (2n-1)(2n-3)} \frac{(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}}$$

$\therefore$ co-efficient of h^n in the expansion

$$= \frac{1.3.5.(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} \cdots \right]$$

$$= P_n(x)$$

Recurrence formula for $P_n(x)$

(i) $n P_n(x) = (2n-1)x P_{n-1}(x) - (n-1)P_{n-2}(x)$

Let $v = (1-2xh+h^2)^{-1/2}$ — ①

$$v = \sum_{n=0}^{\infty} h^n P_n(x)$$

$$v^2 = (1-2xh+h^2)^{-1}$$

$$\Rightarrow v^2(1-2xh+h^2) = 1$$

Differentiating w.r.t h

$$2v\frac{dv}{dh}(1-2xh+h^2) + v^2(2h-2x) = 0$$

$$\Rightarrow \frac{dv}{dh}(1-2xh+h^2) + v(h-x) = 0$$

$$\Rightarrow (1-2xh+h^2)\sum_{n=1}^{\infty} n h^{n-1} P_n(x) + (h-x)\sum_{n=0}^{\infty} h^n P_n(x) = 0$$ — ②

In eqn(2) equate the co-efficient of h^{n-1} to zero.

$$n P_n(x) - 2x(n-1)P_{n-1}(x) + (n-2)P_{n-2}(x) + P_{n-2}(x) - xP_{n-1}(x) = 0$$

$$nP_n(x) = 2x(n-1)P_{n-1}(x) + xP_{n-1}(x) - (n-2)P_{n-2}(x) + P_{n-2}(x)$$

$$= (2n-1)x\,P_{n-1}(x) - (n-1)P_{n-2}(x) \quad \text{--- (A)}$$

Equate the co-efficients of h^n in (2) to zero,

$$(n+1)P_{n+1}(x) - 2xn\,P_n(x) + (n-1)P_{n-1}(x) + P_{n-1}(x) - xP_n(x) = 0$$

$$\Rightarrow 2nx\,P_n(x) + xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

$$\Rightarrow (2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \quad \text{--- (B)}$$

Consider $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Differentiate w.r.t x

$$\frac{h}{(1-2xh+h^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)h^n$$

$$(1-2xh+h^2)\sum_{n=0}^{\infty} P_n'(x)h^n = h(1-2xh+h^2)^{-1/2}$$

$$= h\sum_{n=1}^{\infty} P_n(x)h^n$$

Equate the co-efficient of h^n on both sides of the equation

$$P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x) = P_{n-1}(x)$$

Replace n by $n+1$

$$P_{n+1}'(x) - 2xP_n'(x) + P_{n-1}'(x) = P_n(x)$$

$$P_{n+1}'(x) + P_{n-1}'(x) = 2xP_n'(x) + P_n(x) \quad \text{--- (C)}$$

We have,

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \quad \text{--- (B)}$$

Differentiate (B) w.r.t. x.

$$(2n+1)P_n(x) + (2n+1)xP_n'(x) = (n+1)P_{n+1}'(x) + nP_{n-1}'(x)$$

Multiply both sides by (2) and add $(2n+1)$ times eqn (C)

$$2(2n+1)P_n(x) + 2(2n+1)P_n'(x) + (2n+1)P_{n+1}'(x) + (2n+1)P_{n-1}'(x) = 2(n+1)P_{n+1}'(x) + 2nP_{n-1}'(x) + (2n+1)2xP_n'(x) + (2n+1)P_n(x)$$

$$2(2n+1)P_n(x) - (2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \quad \text{—(8)}$$

Rodrigue's formula :-

$$P_n(x) = \frac{1}{2^n \lfloor n}\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

Let $y = (x^2-1)^n$

$$\frac{dy}{dx} = n(x^2-1)^{n-1}\, 2x = ny\frac{2x}{x^2-1}$$

$$\Rightarrow (x^2-1)\frac{dy}{dx} = 2nyx \quad \text{—(a)}$$

$$\frac{n+1!}{2!\,(n-1)!}$$

$$\frac{(n+1)\,n\,(n-1)!}{(n-1)!}$$

Differentiate (a) (n+1) times by Leibnitz th^m -

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + {}^{n+1}c_1\,(2x)\frac{d^{n+1}}{dx^{n+1}}y + {}^{n+1}c_2\cdot 2\cdot\frac{d^ny}{dx^n}$$

$$= 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}c_1\frac{d^ny}{dx^n}\right]$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + (n+1)(2x)\frac{d^{n+1}y}{dx^{n+1}} + \frac{(n+1)!\,2}{2!\,(n-1)!}\frac{d^ny}{dx^n}$$

$$= 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2n(n+1)\frac{d^ny}{dx^n}$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2x\frac{d^{n+1}y}{dx^{n+1}} + (n+1)n\frac{d^ny}{dx^n}$$

$$= 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2n(n+1)\frac{d^ny}{dx^n}$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\frac{d^ny}{dx^n} = 0$$

putting $\frac{d^ny}{dx^n} = z$ we get,

$$(x^2-1)\frac{d^2z}{dx^2} + 2x\frac{dz}{dx} - n(n+1)z = 0$$

this is Legendre's eq^n one of whose solution is z.
It is also satisfied by cz or $c\frac{d^ny}{dx^n}$ (c is an arbitrary constant)

$$\therefore P_n(x) = c\frac{d^ny}{dx^n}$$

$$= c\left(\frac{d}{dx}\right)^n (x^2-1)^n$$

$$P_n(x) = c\left(\frac{d}{dx}\right)^n \left[(x+1)^n (x-1)^n\right]$$

$= c\left[n!\,(x+1)^n +\right.$ terms containing $(x-1)$ as one of the factors$\left.\right]$

by Leibnitz thm.

$P_n(1) = c\, 2^n \lfloor n = 1$ $\quad\because P_n(1) = 1$

$$\therefore c = \frac{1}{2^n \lfloor n}$$

$$\therefore P_n(x) = \frac{1}{2^n \lfloor n}\left(\frac{d}{dx}\right)^n (x^2-1)^n \quad \text{Proved //}$$

<u>Orthogonal properties of Legendre's Polynomial</u> To show that

$$\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0 \quad \text{if } m \neq n$$

$$= \frac{2}{2n+1} \quad \text{if } m = n$$

$$= \frac{2}{2n+1}\delta_{mn}, \quad m, n \text{ are +ve integers.}$$

<u>Proof:-</u> We have, from Rodrigue's formula -

$$P_n(x) = \frac{1}{2^n \lfloor n}\left(\frac{d}{dx}\right)^n (x^2-1)^n.$$

Now

$$\int_{-1}^{+1} P_m(x) P_n(x)\,dx$$

$$= \frac{1}{2^{m+n}\lfloor m \lfloor n}\int_{-1}^{+1}\left(\frac{d}{dx}\right)^m (x^2-1)^m \left(\frac{d}{dx}\right)^n (x^2-1)^n dx$$

$$= \frac{1}{2^{m+n}\lfloor m \lfloor n}\left\{\left[\left(\frac{d}{dx}\right)^m (x^2-1)^m \left(\frac{d}{dx}\right)^{n-1}(x^2-1)^n\right]_{-1}^{+1} - \int_{-1}^{+1}\left(\frac{d}{dx}\right)^{m+1}(x^2-1)^m \left(\frac{d}{dx}\right)^{n-1}(x^2-1)^n dx\right\}$$

$$= \frac{-1}{2^{m+n}\lfloor m \lfloor n}\int_{-1}^{+1}\left(\frac{d}{dx}\right)^{m+1}(x^2-1)^m \left(\frac{d}{dx}\right)^{n-1}(x^2-1)^n dx$$

Continuing the process of integration by parts (m) times on the R.H.S we get,

$$\int_{-1}^{+1} P_m(x) P_n(x)dx = \frac{(-1)^m}{2^{m+n} \lfloor m \lfloor n} \int_{-1}^{+1} \left(\frac{d}{dx}\right)^{2m} (x^2-1)^m \left(\frac{d}{dx}\right)^{n-m} (x^2-1)^n dx$$

$$\left(\frac{d}{dx}\right)^{2m} (x^2-1)^m = \frac{d^{2m}}{(dx)^{2m}} (x^2-1)^m = 2m!$$

$$\therefore \int_{-1}^{+1} P_m(x) P_n(x)dx = \frac{(-1)^m \lfloor 2m}{2^{m+n} \lfloor m \lfloor n} \int_{-1}^{1} \left(\frac{d}{dx}\right)^{n-m} (x^2-1)^n dx$$

$$= \frac{(-1)^m \lfloor 2m}{2^{m+n} \lfloor m \lfloor n} \left| \left(\frac{d}{dx}\right)^{n-m-1} (x^2-1)^n \right|_{-1}^{+1}$$

$= 0$ for both the limits when $n \neq m$, $n > m$

When $n = m$,

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx = \int_{-1}^{+1} P_n^2(x) dx$$

$$= \frac{(-1)^n \lfloor 2n}{2^{2n} \lfloor n \lfloor n} \int_{-1}^{+1} (x^2-1)^n dx$$

$$= \frac{(-1)^n\, 1.2.3. \ldots 2n\, (-1)^n}{2^{2n} \lfloor n \lfloor n} \int_{-1}^{+1} (1-x^2)^n dx$$

$$= \frac{(-1)^{2n}\, 1.3.5. \ldots (2n-1)\, 2^n \lfloor n}{2^{2n} \lfloor n \lfloor n} \int_{-1}^{+1} (1-x^2)^n dx$$

$$= \frac{1.3.5 \ldots (2n-1)}{2^n \lfloor n} \int_{-1}^{+1} (1-x^2)^n dx$$

Putting $x = \cos\theta$, $dx = -\sin\theta\, d\theta$

$$(1-x^2)^n = (1-\cos^2\theta)^n$$
$$= (\sin^2\theta)^n$$
$$= \sin^{2n}\theta$$

$$\therefore \int_{-1}^{+1} (1-x^2)^n dx$$

$x = -1 = \cos\theta$
$\Rightarrow \theta = \pi$

$x = 1 = \cos\theta$
$\Rightarrow \theta = 0$

$$= -\int_{\pi}^{0} \sin^{2n}\theta \sin\theta\, d\theta$$

$$= -\int_{\pi}^{0} \sin^{2n+1}\theta\, d\theta$$

$$\therefore \int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = -\frac{1.3.5\ldots(2n-1)}{2^n\, n!} \int_{\pi}^{0} \sin^{2n+1}\theta \, d\theta$$

$$= \frac{1.3.5\ldots(2n-1)\, 2}{2^n\, n!} \int_{0}^{\pi/2} \sin^{2n+1}\theta \, d\theta$$

$$= \frac{2.1.3.5\ldots(2n-1)}{2^n\, n!} \cdot \frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3.1}$$

$$= \frac{2.2^n . n!}{2^n\, n!\,(2n+1)} = \frac{2}{2n+1}$$ Proved //