Complex Nos. Complex no. is written in the form a+ib where a & b are real nos and $i=\sqrt{-1}$ is called the imaginary unit. a is called the real part and b imaginary part of z. If a=0 then z = ib is called pure imaginary. Two complex no.s a+ib & c+id are equal if and only a=c, b=d. If z denotes any one of a set of complex nos then z is called a complex variable.

If a+ib is a complex no then a−ib is called complex conjugate of a+ib. For a complex no. z the complex conjugate is denoted by z^* or $\bar{z}$.

For a complex no. $z = a+ib$, the quantity $\sqrt{a^2+b^2}$ is called the modulus of z and is denoted by $|z|$.

Algebraic operations:-

If $z_1 = a+ib$, $z_2 = c+id$

1) Addition :- $z_1+z_2 = (a+c)+i(b+d)$
2) Substraction :- $z_1-z_2 = (a-c)+i(b-d)$
3) Multiplication :- $z_1z_2 = (a+ib)(c+id) = (ac-bd)+i(bc+ad)$
4) Division :- $z_1/z_2 = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{(bc-da)}{c^2+d^2}$

Q) Show that $|z_1z_2| = |z_1||z_2|$

Soln:- Let $z_1 = a+ib$ and $z_2 = c+id$.

$z_1z_2 = (ac-bd)+i(bc+ad)$

$|z_1z_2| = \sqrt{(ac-bd)^2+(bc+ad)^2}$

$= \sqrt{a^2c^2+b^2d^2-2abcd+b^2c^2+a^2d^2+2abcd}$

$= \sqrt{(a^2+b^2)(c^2+d^2)}$

$|z_1| = \sqrt{a^2+b^2}$

$|z_2| = \sqrt{c^2+d^2}$

$\therefore |z_1||z_2| = \sqrt{(a^2+b^2)(c^2+d^2)}$

$= |z_1z_2|$

$|z|^2 = \bar{z}\cdot z$

$\Rightarrow |z|^2 = \bar{z}z$

$|z_1z_2|^2 = z_1z_2\overline{z_1z_2}$

$= z_1z_2\bar{z}_1\bar{z}_2$

$= z_1\bar{z}_1 z_2\bar{z}_2$

$= |z_1|^2|z_2|^2$

$\therefore |z_1z_2| = |z_1||z_2|$

$|z|^2 = \bar{z}z$

$|z_1z_2|^2 = z_1z_2\overline{z_1z_2}$

$= |z_1|^2|z_2|^2$

$|z_1z_2| = |z_1||z_2|$

$|z_1z_2|^2 = z_1z_2\overline{z_1z_2}$

$= z_1z_2\bar{z}_1\bar{z}_2$

$= |z_1|^2|z_2|^2$

$|z_1z_2| = |z_1||z_2|$

Q) Show that $|z_1+z_2| \le |z_1|+|z_2|$

Pf

$z_1 = x_1+iy_1 \quad z_2 = x_2+iy_2$

$|(x_1+x_2)+i(y_1+y_2)| \le |x_1+iy_1|+|x_2+iy_2|$

$\sqrt{(x_1+x_2)^2+(y_1+y_2)^2} \le \sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}$

$(x_1+x_2)^2+(y_1+y_2)^2 \le x_1^2+y_1^2+x_2^2+y_2^2+2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$

$x_1^2+x_2^2+2x_1x_2+y_1^2+y_2^2+2y_1y_2 \le x_1^2+y_1^2+x_2^2+y_2^2+2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$

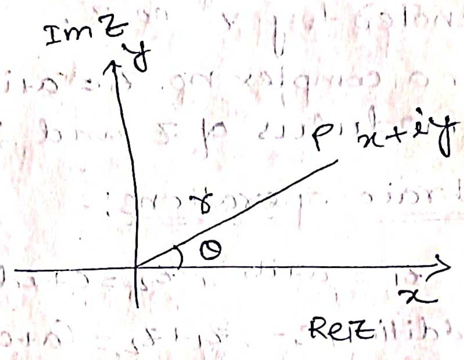
$x_1x_2+y_1y_2 \le \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2 \le x_1^2x_2^2+x_1^2y_2^2+y_1^2x_2^2+y_1^2y_2^2$

$x_1^2y_2^2+x_2^2y_1^2-2x_1y_1x_2y_2 \ge 0$

$(x_1y_2-x_2y_1)^2 \ge 0$

Polarform :-



If P is a pt. on the complex plane corresponding to the complex no $x+iy$, then -

$x = r\cos\theta, \quad y = r\sin\theta$

where $r = \sqrt{x^2+y^2} = |x+iy|$ is called the modulus or the absolute value of $z = x+iy$ and θ is called the argument or amplitude of z. It follows that $z = r(\cos\theta + i\sin\theta)$ and is called the polar form of complex no. z. Also we have, $\tan\theta = y/x$

Q) Write the following complex no. in polar form -

1. $2+2\sqrt{3}i$

$r = \sqrt{4+12}$
$= 4$

$\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$\therefore 2+2\sqrt{3}i = 4(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

2. $-5+5i$

$r = \sqrt{25+25}$
$= 5\sqrt{2}$

$\tan\theta = \frac{5}{-5} = -1$

$\theta = \frac{3\pi}{4}$

$\therefore -5+5i = 5\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$

3. $\sqrt{6}-\sqrt{2}i$

$r = \sqrt{6+2}$
$= 2\sqrt{2}$

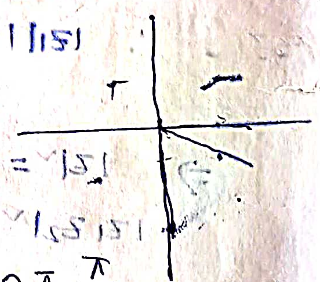
$\tan\theta = \frac{\sqrt{2}}{\sqrt{6}}$

$= \frac{\sqrt{2}}{\sqrt{2}\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

$\therefore 2\sqrt{2}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$



$= 2\pi - \frac{\pi}{6}$

$= \frac{11\pi}{6}$

$\pi - \pi/4$

$= 3\pi/4$

De Moivre's theorem:-

If n is a +ve integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof:- If $n=1$ $\quad (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

$n=2 \quad (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

$= \cos 2\theta + i\sin 2\theta$

Hence the th^m holds for $n=1, 2$

Assume that it holds for $n=k$, k being some +ve integer.

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Multiplying both side of the eq above by $(\cos\theta + i\sin\theta)$

$(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$\Rightarrow (\cos\theta + i\sin\theta)^{k+1} = \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

Since, it holds for $n = k+1$, If it is true for $n=k$.

Hence by induction, it holds for all +ve integers.

De Moivre's th^m also holds for n, a -ve integer.

Let $n = -m$ where m is a +ve integer.

$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$= \dfrac{1}{(\cos\theta + i\sin\theta)^m}$

$= \dfrac{1}{\cos m\theta + i\sin m\theta}$

$= \dfrac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$

$= \dfrac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$

$= \cos m\theta - i\sin m\theta$

$= \cos(-n\theta) - i\sin(-n\theta)$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\cos(2\pi - \pi/6) + i\sin(2\pi - \pi/6)$

$= \cos\pi/6 - i\sin\pi/6$

$\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$\sqrt{6} - \sqrt{2}$

$=$

$\sin\theta =$

If n is a rational no. i.e $n = p/q$, where p & q are integers, then $\cos \frac{p}{q}\theta + i\sin \frac{p}{q}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$

we have, $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^{q} = \cos\theta + i\sin\theta$

Taking the q^{th} root of both sides we get that one of the q^{th} roots of $(\cos\theta + i\sin\theta)$ is $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$

i.e $(\cos\theta + i\sin\theta)^{1/q} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$

Taking p^{th} power of both sides –

$$(\cos\theta + i\sin\theta)^{p/q} = \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^{p}$$

Using De Moivre's th^m –

$$(\cos\theta + i\sin\theta)^{p/q} = \cos \tfrac{p}{q}\theta + i\sin \tfrac{p}{q}\theta$$

Roots of complex Numbers :-

If ω and z are complex nos such that $\omega^n = z$ where n is a +ve integer then, $\omega = z^{1/n}$ is a complex root of z.

If $z = r(\cos\theta + i\sin\theta)$, then,

$$\omega = z^{1/n} = r^{1/n}\left[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right]^{1/n}$$

$$= r^{1/n}\left[\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right]$$

integrated $k = 0, 1, 2, 3, \ldots (n-1)$

Q) Find each of the root and locate them graphically.

1.(a) $(2 + i2\sqrt{3})^{1/3}$

$z = 2 + 2\sqrt{3}$

$r = \sqrt{4+12}$

$= 4$

$\tan\theta = \frac{2\sqrt{3}}{2}$

$= \sqrt{3}$

$\theta = \frac{\pi}{3}$

$\therefore z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

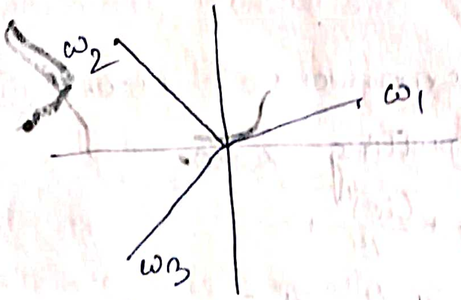
$$\omega_n = 4^{1/3}\left[\cos\frac{\pi/3 + 2k\pi}{3} + i\sin\frac{\frac{\pi}{3} + 2k\pi}{3}\right] \quad k = 0, 1, 2$$

$\omega_1 = 4^{1/3}\left[\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right]$

$\omega_2 = 4^{1/3}\left[\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right]$

$\omega_3 = 4^{1/3}\left[\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right]$

① $1^{1/3}$ ② $i^{1/3}$ ③ $(-1)^{1/3}$



b) $(-1+i)^{1/3}$ Let $z=-1+i$

$r=\sqrt{1+1}=\sqrt{2}$

$\tan\theta=\frac{1}{-1}=-1$

$\theta=\tan^{-1}(-1)$

$=\frac{3\pi}{4}$

$\therefore z=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

$w_n=(\sqrt{2})^{1/3}\left(\cos\frac{\frac{3\pi}{4}+2k\pi}{3}+i\sin\frac{\frac{3\pi}{4}+2k\pi}{3}\right),\ k=0,1,2$

$w_1=(\sqrt{2})^{1/3}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$w_2=(\sqrt{2})^{1/3}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$

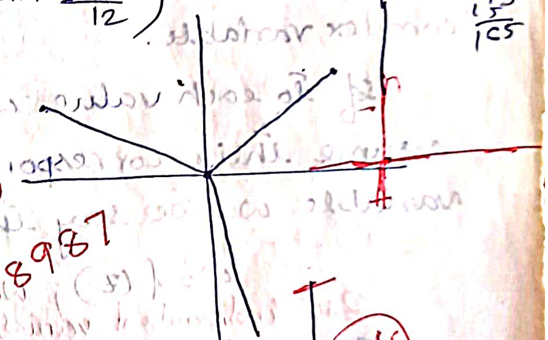
$w_3=(\sqrt{2})^{1/3}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$

$90°+45°$, $\frac{\pi}{2}+\frac{\pi}{4}$, $=\frac{2\pi+\pi}{4}$, $\frac{2\pi}{12}$

$\frac{\frac{3\pi}{4}+2\pi}{3}=\frac{11\pi}{4\times3}$, $\frac{3\pi}{4}+4\pi$

$\frac{11\times180}{12}$

1539
2913



8473943071
9707508987

Fail

c) $(-32)^{1/5}$

$z=-32+i\cdot0$

$r=\sqrt{\quad}\ 32$

$\theta=\tan^{-1}(0)$

$=0$

$z=32(\cos 0+i\sin 0)$

$w^n=32^{1/5}\left(\cos\frac{2k\pi}{5}\right)$

$\theta=\pi$

$z=32(\cos\pi+i\sin\pi)$

$w^n=(32)^{1/5}\left(\cos\frac{\pi+2k\pi}{5}+i\sin\frac{\pi+2k\pi}{5}\right)$

$=2\left(\cos\frac{\pi+2k\pi}{5}+i\sin\frac{\pi+2k\pi}{5}\right)$

$k=0,1,2,3,4$

$w_1=2\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$

$w_2=2\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$

$w_3=2\left(\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}\right)$

$w_4=2\left(\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}\right)$

$w_5=2\left(\cos 9\pi/5+i\sin 9\pi/5\right)$

Euler's formulae :- (1)

If we assume that the expansion $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} \dots$ holds for x complex, then putting $x = i\theta$.

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{\lfloor 2} + \frac{(i\theta)^3}{\lfloor 3} + \frac{(i\theta)^4}{\lfloor 4} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{\lfloor 2} - \frac{i\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} + \dots$$

$$= \left(1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} \dots\right) + i\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} \dots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

• Alternatively one may define the complex function $e^z = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i\sin y)$

Hyoid bone.

$e^{i\pi} + 1 = 0$

Functions, limit and continuity :-

Variable and Function: A symbol such as z which can represent anyone of a set of complex no is called a complex variable.

If to each value which a complex variable z can assume their corresponds one or more values of a complex variable ω, we say that ω is funcn of z.

$\omega = f(z)$, $\omega = g(z)$ etc.

z is independent variable and ω is dependent variable.

The value of a funcn at $z = z_0$ is written as $f(z_0)$

If only one value of ω corresponds to each value of z, we say that ω is a single valued funcn of z. If more than one value of ω corresponds each value of z, then ω is called a multiple valued funcn of z.

If $\omega = u + iv$, where u & v are real, is a single valued funcn of $z = x + iy$, where x & y are real, then we can write $u + iv = f(x + iy)$

Equating real & imaginary part $u = u(x, y)$
$v = v(x, y)$

* Single valued funcn: Ex. If $\omega = z^2$, then to each value of z there is only one value of ω. Hence $\omega = f(z) = z^2$ is a single valued funcn.
[Mul]tiple valued funcn:- Ex. If $\omega = z^{1/2}$, then to each value of z

Elementary functions :-

1. Polynomial funcⁿs can be defined by

$$w = a_0 z^n + a_1 z^{n-1} + \dots + a_n = P(z), \text{ where } a_0 \neq 0$$

and $a_0, a_1 \dots$ etc are complex constants and n is a +ve integer, which is called degree of the polynomial. The funcⁿ $w = az+b$ is called linear transformation.

2. Rational Algebraic ~~funcⁿ fractions~~ funcⁿ are defined by

$w = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are ~~+ve~~ polynomials. The special case $w = \frac{az+b}{cz+d}$ with $ad-bc \neq 0$ is called bi-linear or fractional linear transformation.

3. Exponential functions are defined by $w = e^z = e^{x+iy}$ $= e^x e^{iy} = e^x(\cos y + i\sin y)$ where $e = 2.71828\dots$ is the base of natural logarithm. If a is real & +ve, we define $a^z = e^{z \ln a}$

4. Trigonometric functions are defined by $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ etc.

$$\sin^2 z + \cos^2 z = 1$$

$$\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2$$

$$= \frac{e^{2iz} + e^{-2iz} - 2}{-4} + \frac{e^{2iz} + e^{-2iz} + 2}{4}$$

$$= \frac{-e^{2iz} - e^{-2iz} + 2 + e^{2iz} + e^{-2iz} + 2}{4}$$

$$= 1$$

$$\cos i = \frac{e^{i \cdot i} + e^{-i \cdot i}}{2}$$

$$= \frac{e^{-1} + e^{1}}{2}$$

$$= \frac{\frac{1}{e} + e}{2}$$

$$= \frac{1+e^2}{2e} > 1$$

if $f(z) = z^2$ then $f(2i) = (2i)^2 = -4$

① $\sqrt{i}$

②

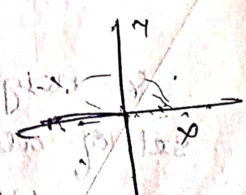
$$\left(-2\sqrt{3} - 2i\right)^{1/4}$$

$$r = \sqrt{12+4} = 4$$

$$\tan\theta = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\frac{\pi}{6}$$

$$\pi + \pi/6 = 7\pi/6$$



③ $\left(-2\sqrt{3} + 2i\right)^{1/4}$

$$\tan\theta = -\frac{1}{\sqrt{3}} = \tan\left(\pi - \frac{\pi}{6}\right)$$

$$= \frac{5\pi}{6}$$

5. Hyperbolic functions

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

Show that $\cosh^2 z - \sinh^2 z = 1$

$1 - \tanh^2 z = \operatorname{sech}^2 z$

$\coth^2 z - 1 = \operatorname{cosech}^2 z$

$$\left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$$

$$= \frac{e^{2z} + e^{-2z} + 2}{4} - \left\{\frac{e^{2z} + e^{-2z} - 2}{4}\right\}$$

$$= 1$$

$\sin iz = i\sinh z$, $\cos iz = \cosh z$, $\tan iz = i\tanh z$

$\sinh iz = i\sin z$, $\cosh iz = \cos z$, $\tanh iz = i\tan z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\sin iz = \frac{e^{-z} - e^{z}}{2i}$$

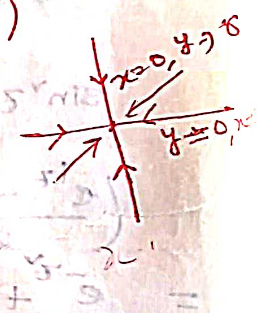
$$= -i\,\frac{(e^{-z} - e^{z})}{2}$$

$$= i\left(\frac{e^z - e^{-z}}{2}\right)$$

$$= i\sinh z$$

$$\lim_{z\to 0} \frac{|z|^2}{Re(z^2)}$$

$x \to 0$, $y \to 0$



Limit

$$|z - z_0| = r$$

δ-neighbourhood: δ-neighbourhood of a pt z_0 is the set of all points z such that $|z - z_0| < \delta$, where δ is any +ve no.

A deleted δ neighbourhood of z_0 is a neighbourhood of z_0 in which the point z_0 is omitted i.e

$$0 < |z - z_0| < \delta$$

Limit Let $f(z)$ be defined and single valued in a deleted δ-neighbourhood of $z = z_0$. We then say that the no. l is the limit of $f(z)$ as z approaches z_0

as $\lim_{z\to z_0} f(z) = l$

If for any +ve no. (however small) ∈ we can find some +ve no. δ such that, $|f(z)-l|<\epsilon$ whenever $0<|z-z_0|<\delta$.

This limit must be independent of the manner in which z approaches z_0.

Q) Show that $\lim_{z\to 0} \frac{z^*}{z}$ does not exist.

⇒ $\lim_{z\to 0} \frac{z^*}{z}$

$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy}$

If we take the limit as $z\to 0$ along the x-axis.

$\Rightarrow \lim_{x\to 0} \frac{x}{x}$

$\Rightarrow \lim_{x\to 0} 1$ $\quad x=0$

$= 1$

along y-axis

$\lim_{y\to 0} \frac{-iy}{+iy}$

$= \lim_{y\to 0} (-1)$

$= (-1)$

$\therefore \lim_{z\to 0} \frac{z^*}{z}$ doesnot exist.

Continuity :- Let f(z) be defined and single valued in a δ-neighbourhood of $z=z_0$. The func^n f(z) is said to be continuous at $z=z_0$ if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

The above implies three cond^ns.

① $\lim_{z\to z_0} f(z) = l$ must exist

② $f(z_0)$ must exist

③ $l = f(z_0)$

$\lim_{z\to -1} z^{1/2}$ $\quad Z=|z|e^{i\theta}$

$|z|=1$

$\theta\to\pi$

Transformations

$w=z^2$

$u+iv=(x+iy)^2$

$=x^2-y^2+2ixy$

$u=x^2-y^2$, $v=2xy$

$i^2=2$
$=-3$

$\frac{x^2-y^2+2ixy}{x+iy+1}$

$\lim_{z\to -i}$ $\lim_{y\to -1} \frac{-y^2}{-iy+1}$

inverse functions

$\omega=f(z)$ then z can be considered as a function of ω.

$z=g(\omega)=f^{-1}(\omega)$

$\lim_{y\to -1} \frac{-y^2}{-iy+1}$

$= -\frac{1}{i+1}$

Derivatives :- If $f(z)$ is single valued in some region R of the complex z-plane, the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$ provided that this limit exist independent of the manner in which $\Delta z \to 0$.

In such a case we say that $f(z)$ is differentiable at z.

Analytic funcn : If the derivative $f'(z)$ exist at all pts z of a region R, then $f(z)$ is said to be analytic in R and is referred to as a funcn analytic at R. Analytic funcns are also called regular or holomorphic.

A funcn $f(z)$ is said to be analytic at a pt z_0 if there exist a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exist.

Q) Show that $\frac{d}{dz} z^*$ doesn't exist anywhere.

⇒ $z = a+ib$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^* - z^*}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\Delta z)^*}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Taking the limit parallel to the x-axis, $\Delta y \to 0$

$$f(z) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

$$= 1$$

Taking the limit parallel to the y-axis, $\Delta x \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y}$$

$$= -1$$

∴ $\frac{d}{dz} z^*$ doesn't exist.

Q) Show that if $f(z)$ is analytic at z_0, then it must be continuous at z_0.

$\Rightarrow$ $f(z_0+\Delta z) - f(z_0) = \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \Delta z$

$$\lim_{\Delta z \to 0} [f(z_0+\Delta z) - f(z_0)] = \lim_{\Delta z \to 0} \left[\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \cdot \Delta z \right]$$

$$= f'(z_0) \times 0$$

$$= 0$$

$$\lim_{\Delta z \to 0} f(z_0+\Delta z) = f(z_0)$$

Singular points :-

A pt at which a funcn $f(z)$ fails to be analytic, is called a singular pt or singularity of $f(z)$.

1. Isolated singularity : The point $z = z_0$ is called an isolated singularity or an isolated singular pt. of $f(z)$ if we can find some $\delta > 0$ such that the circle $|z - z_0| = \delta$ ~~etc~~ encloses no singular pt other than z_0. That is if there exist a deleted δ neighbourhood of z_0 which contains no singularity.

2. Pole :- If we can find a +ve ~~no.~~ integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called a pole of order n. If $n = 1$, then z_0 is called a simple pole.

3. Removable singularity :- A singular pt z_0 is called a removable singularity of $f(z)$ if,

$$\lim_{z \to z_0} f(z) \text{ exists.}$$

4. essential singularity :- The singularity which is not a pole, a branch pt or a removable singularity is called an essential singularity.

The Cauchy - Riemann eqn

Theorem :- A necessary cond'n that $w = f(z) = u(x,y) + i\,v(x,y)$ be analytic in a region R is that, in R, u & v satisfy the Cauchy - Riemann eq'ns $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ - If the partial derivatives in the above relations are continuous in R; the Cauchy - Riemann eqns are a set of sufficient conditions that f(z) be analytic in R.

Proof :-

A) The equations are necessary :-

Let f(z) be an analytic funcn of z and let,

$$f(z) = u(x,y) + i v(x,y)$$

$$\therefore \partial f = \partial f(z) = \partial u + i \partial v$$

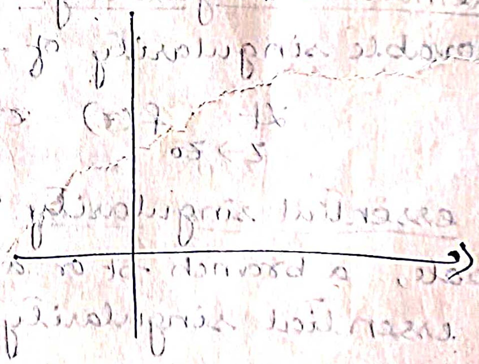
Again, $\partial z = \partial x + i \partial y$

$$\frac{\partial f}{\partial z} = \frac{\partial u + i \partial v}{\partial x + i \partial y}$$

According to our assumption

$$f'(z) = \lim_{\partial z \to 0} \frac{\partial f}{\partial z} = \lim_{\substack{\partial x \to 0 \\ \partial y \to 0}} \frac{\partial u + \partial \partial v}{\partial x + i \partial y} \text{ exist}$$

Therefore this limit must be independent of the manner in which $\partial x \to 0$ & $\partial y \to 0$.



If we approach the limit ll to the x - axis, then $\partial y = 0$ & $\partial x \to 0$ so that $f'(z) = \lim_{\partial x \to 0} \frac{\partial u + i \partial v}{\partial x}$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{— (1)}$$

Again taking the limit || to the y axis, we have,
$\partial x = 0$, $\partial y \to 0$, so that

$$f(z) = \lim_{\partial y \to 0} \frac{\partial u + i \partial v}{i \partial y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \text{--- (2)}$$

Since the limit exists, equating (1) & (2)

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are Cauchy-Riemann eqn. Hence the condn is necessary.

(B) The eqns are sufficient provided the partial derivatives of u & v are continuous.

Let f(z) be a single valued funcn of z, in some domain R

$$\delta f(z) = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$$

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right) \delta y \quad \text{--- (3)}$$

The above eqn can be written, since it is assumed that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous in the domain. Therefore

$$\frac{\delta f}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right) \delta y}{\delta x + i \delta y}$$

$$= \frac{\left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right) \frac{\delta y}{\delta x}}{1 + i \frac{\delta y}{\delta x}}$$

using Cauchy-Riemann eqns

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ to replace

y-derivatives, we get

$$\frac{\delta f}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}}$$

$$\frac{\delta f}{\delta z} = \frac{\frac{\partial u}{\partial x}\left(1 + i\frac{\delta y}{\delta x}\right) + i\frac{\partial v}{\partial x}\left(1 + i\frac{\delta y}{\delta x}\right)}{1 + i\frac{\delta y}{\delta x}}$$

$$= \frac{\left(1 + i\frac{\delta y}{\delta x}\right)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)}{1 + i\frac{\delta y}{\delta x}}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

If we take the limit $\delta z \to 0$, the value will be independent of the manner in which $\delta x \to 0$ & $\delta y \to 0$. Therefore the limit exists and hence the derivative exists. Therefore $f(z)$ is an analytic funcⁿ.

Q) Show that e^z is analytic.

⇒ Here, $f(z) = e^z$

$= e^{x+iy}$

$= e^x . e^{iy}$

$= e^x (\cos y + i \sin y)$

$f(z) = e^x (\cos y + i \sin y)$

$$\frac{\partial u}{\partial x} = \cos y \, e^x$$

$$\frac{\partial u}{\partial y} = e^x(-\sin y) = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = \sin y \, e^x$$

$$\frac{\partial v}{\partial y} = e^x \cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ e^z is analytic

Use the Cauchy - Riemann equations to find out whether the following funcⁿs are analytic.

1. z^3
2. $\sin z$
3. $\cosh z$
4. e^{iz}
5. $\frac{x-iy}{x^2+y^2}$
6. $\frac{y-ix}{x^2+y^2}$

soln:- ① $f(z) = z^3$

$= (x+iy)^3$

$= x^3 - iy^3 + 3xiy(x+iy)$

$= x^3 - iy^3 + 3x^2iy - 3xy^2$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$u = x^3 - 3xy^2 \qquad v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore z^3$ is analytic.

② $f(z) = \sin z$

$= \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh z$

$u = \sin x \cosh y \qquad v = \cos x \sinh z$

$u = \sin x \cdot \frac{e^y + e^{-y}}{2}$

$\frac{d}{dz}\sinh z = \cosh z$

$\frac{d}{dz}\cosh z = \sinh z$

$\frac{\partial u}{\partial x} = \cosh y \cos x$

$\frac{\partial u}{\partial y} = \sin x \sinh z$

$\frac{\partial v}{\partial y} = \cos x \cosh z$

$\frac{\partial v}{\partial x} = -\sin x \sinh z$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore \sin z$ is analytic.

4. e^{iz}

$\Rightarrow f(z) = e^{iz}$

$= e^{i(x+iy)}$

$= e^{ix-y}$

$= e^{ix} \cdot e^{-y}$

$= (\cos x + i \sin x) e^{-y}$

$= e^{-y}\cos x + i e^{-y} \sin x$

$u = \cos x \, e^{-y} \qquad v = e^{-y} \cdot \sin x$

$\frac{\partial u}{\partial x} = -e^{-y} \sin x \qquad \frac{\partial u}{\partial y} = \cos x (-e^{-y})$

$\frac{\partial v}{\partial x} = e^{-y} \cos x \qquad \frac{\partial v}{\partial y} = \sin x (-e^{-y})$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore$ e^{iz} is analytic.

5. $\frac{x-iy}{x^2+y^2}$

$f(z) = \frac{x-iy}{x^2+y^2}$

$= \frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2}$

$u = \frac{x}{x^2+y^2} \qquad v = -\frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

$= \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{0-2xy}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial x} = -\left[\frac{0-2xy}{(x^2+y^2)^2}\right] = \frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial y} = -\left[\frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2}\right]$

$= -\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$

$= \frac{y^2-x^2}{(x^2+y^2)^2}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore \frac{x-iy}{x^2+y^2}$ is analytic.

6.

Cauchy-Rimann eqns in polar co-ordinates :-

We have the following transformation from Cartesian to plane polar co-ordinates.

$$x = r\cos\theta, \quad y = r\sin\theta \quad -(1)$$

The real & imaginary part of a complex funn $w = f(z) = u + iv$; u & v are funcn of x & y or r & θ

Using the chain rule -

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad -(2)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \quad -(3)$$

In this case using eqn (1), (2) & (3) can be written as

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\ \frac{\partial u}{\partial \theta} &= -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta \end{aligned}\right\} \quad -(4)$$

Similarly for the funcn v, we have -

$$\left.\begin{aligned} \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\ \frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta \end{aligned}\right\} \quad -(5)$$

Using the cauchy-Riemann eqns in eqn (5)

$$\left.\begin{aligned} \frac{\partial v}{\partial r} &= -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta \\ \frac{\partial v}{\partial \theta} &= \frac{\partial u}{\partial y} r\sin\theta + \frac{\partial u}{\partial x} r\cos\theta \end{aligned}\right\} \quad -(6)$$

Comparing eqn (6) & (4)

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}\right\} \quad -(7)$$

Eqn (7) are Cauchy-Riemann eqns in polar co-ordinate.

Harmonic Functions:-

Any real valued funcn ϕ of two real variables x & y is said to be harmonic in a given domain of the x-y plane, if throughout the domain, it has continuous partial derivatives of the 1st and 2nd order, and satisfies the partial differential eqn - $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ —(1)

Eqn (1) is called Laplace's eqn.

Theorem 1:- If a funcn $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component funcns u and v are harmonic.

Proof:- The funcn is analytic. So,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—(1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—(2)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}$$

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\therefore \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

CR eqns with z and z^* as the independent variables:

Since $z = x+iy$ and $z^* = x-iy$, we can express a funcn in terms of z and z^* as the independent variable. Now,

$$x = \frac{1}{2}(z+z^*), \quad y = \frac{1}{2i}(z-z^*)$$

Let $f(z)$ be an analytic funcn in some region R of the complex z plane

$$f(z) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right) + iv\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right)$$

$$= f(z, z^*)$$

$$\frac{\partial f(z,z^*)}{\partial z^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y}$$

$$\frac{\partial f(z,z^*)}{\partial z^*} = \frac{1}{2}\frac{\partial}{\partial x}[u(x,y) + iv(x,y)] - \frac{1}{2i}\frac{\partial}{\partial y}[u(x,y) + iv(x,y)]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] - \frac{1}{2i}\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

Since $f(z)$ is analytic, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial f(z,z^*)}{\partial z^*} = 0$$

Ex Prove that $f(z)$ is analytic in a region R including the point z_0, then $f(z) = f(z_0) + ~~f'(z-z_0)~~ f'(z_0)(z-z_0) + \eta(z-z_0)$

$\eta \to 0$ as $z \to z_0$ | $f(z) =$

$\Rightarrow$

Let $$\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \eta$$

then $$\lim_{z\to z_0}\left[\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right] = \lim_{z\to z_0} \eta$$

$$\therefore \lim_{z\to z_0} \eta = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$$

$$0 = f'(z) - f'(z_0)$$

L' Hospital rule :

Let $f(z)$ and $g(z)$ be analytic in a region containing the pt z_0 and suppose that,

$$f(z_0) = g(z_0) = 0 \quad \text{but } g'(z_0) \neq 0,$$

Then L' Hospital rule states that,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

Proof:- $f(z)$ and $g(z)$ can be written as

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \eta_1(z-z_0)$$

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + \eta_2(z-z_0);$$

where $\eta_1, \eta_2 \to 0$, as $z \to z_0$

Since $f(z_0) = 0$, $g(z_0) = 0$, we have,

$$f(z) = f'(z_0)(z-z_0) + \eta_1(z-z_0)$$

$$g(z) = g'(z_0)(z-z_0) + \eta_2(z-z_0)$$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{(f'(z_0) + \eta_1)(z-z_0)}{(g'(z_0)+\eta_2)(z-z_0)}$$

$$= \frac{f'(z_0)}{g'(z_0)}$$

$z^8 \cdot z$

(Q) $\lim_{z \to i} \dfrac{z^{10}+1}{z^6+1}$

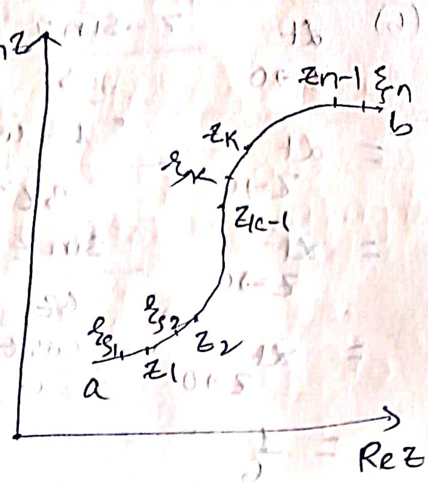
$$= \lim_{z \to i} \frac{10z^9}{6z^5}$$

$$= \frac{10}{6} \lim_{z \to i} \frac{z^9}{z^5}$$

$$= \frac{10}{6}$$

$$= \frac{5}{3}$$

Complex Line Integrals :-



$f(z)$ is be continuous at all pts. of a curve c. Subdivide c into n parts by means of pts $z_1, z_2 \ldots z_{n-1}$ chosen arbitrary and are call

$a = z_0$
$b = z_n$

On each arc joining z_{K-1} to z_K $(K = 1, \ldots, n)$, we choose a pt. ξ_K. Now we form the sum -

$$S_n = f(\xi_1)(z_1 - z_0) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(z_n - z_{n-1})$$

If we write $z_K - z_{K-1} = \Delta z_K$, then the above sum may be written as -

$$S_n = \sum_{K=1}^{n} f(\xi_K)(z_K - z_{K-1}) = \sum_{K=1}^{n} f(\xi_K)\Delta z_K \quad \text{—①}$$

Let the no. of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_K|$ approaches zero. Then sum S_n approaches a limit which doesn't depend on the mode of subdivision and this limit is denoted by

$\int_a^b f(z)dz$ or $\int_C f(z)dz$ which is called the complex line integral of $f(z)$ along the curve C.

If $f(z) = u(x,y) + iv(x,y)$, then the complex line integral can be expressed as -

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int udx - vdy + i\int_C vdx + udy.$$

The above is sometimes taken as a defn of a complex line integral.

Cauchy's thm :-

If $f(z)$ is analytic on and inside a simple closed curve c then, cauchy's thm states that –

$$\oint_c f(z)\,dz = 0$$

Proof:- Let $f(z) = u(x,y) + iv(x,y)$

Then
$$\oint_c f(z)\,dz = \int_c (u+iv)(dx+idy)$$
$$= \int_c u\,dx - v\,dy + i\int_c v\,dx + u\,dy \quad —(A)$$

Green's thm in the plane states that
$$\int_c P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad —(B)$$
where R is the region enclosed by ~~sim~~ simple curve C

using (B) in (A) -
$$\oint_c f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Using Cauchy-Riemann eqn's -
$$\oint_c f(z)\,dz = 0$$

Q) If $f(z)$ is analytic, show that the complex line integral of $f(z)$ betn any two pts a & b is independent of the path joining a & b.

=> $\int_c f(z)\,dz$

By cauchy's thm :-
$$\int_{acbda} f(z)\,dz = 0$$
$$\Rightarrow \int_{acb} f(z)\,dz + \int_{bda} f(z)\,dz = 0$$
$$\Rightarrow \int_{acb} f(z)\,dz = -\int_{bda} f(z)\,dz$$
$$\Rightarrow \int_{acb} f(z)\,dz = \int_{adb} f(z)\,dz$$

∴ The complex line integral of $f(z)$ betn any two pts a & b is independent of the path joining a & b.

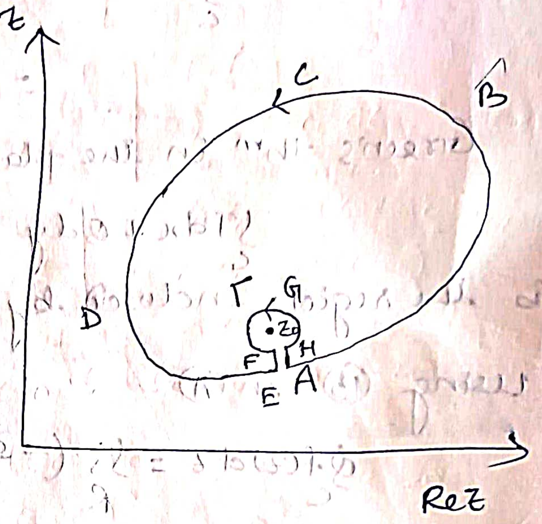
Cauchy's Integral Formula :-

If $f(z)$ is analytic on and inside a simple closed curve c, and z_0 is an internal pt of c, then

$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)\,dz}{z - z_0}$$

where c is traversed in the positive sense.

Proof:-



The interior pt. z_0 is surrounded by a circular contour Γ of radius ϵ and centre at z_0. Γ and c are joined by means of parallel cross-cuts as shown in the diagram.

For the closed curve ABDEFGHA, z_0 is an exterior pt. Using Cauchy's thm -

$$\int_{ABDEFGHA} \frac{f(z)\,dz}{z - z_0} = 0$$

$$= \int_{ABDE} \frac{f(z)\,dz}{z - z_0} + \int_{EF} \frac{f(z)\,dz}{z - z_0} + \int_{FGH} \frac{f(z)\,dz}{z - z_0} + \int_{HA} \frac{f(z)\,dz}{z - z_0}$$

Since the cross-cuts can be brought arbitrarily closed to each other, therefore ....

$$\int_{EF} \frac{f(z)\,dz}{z - z_0} + \int_{HA} \frac{f(z)\,dz}{z - z_0} = 0$$

$$\therefore \int_{ABDE} \frac{f(z)\,dz}{z - z_0} + \int_{FGH} \frac{f(z)\,dz}{z - z_0} = 0$$

$$\oint_c \frac{f(z)\,dz}{z - z_0} + \oint_\Gamma \frac{f(z)\,dz}{z - z_0} = 0$$

where the integral around Γ is taken in the clockwise direcn.

$$\oint_C \frac{f(z)\,dz}{z-z_0} = -\oint_\Gamma \frac{f(z)\,dz}{z-z_0}$$

$$\Rightarrow \oint_C \frac{f(z)\,dz}{z-z_0} = \oint_\Gamma \frac{f(z)\,dz}{z-z_0}$$

Here, both C & Γ are taken in the anti-clockwise direc^n.

The eq^n of the circle Γ is –

$$|z-z_0| = \epsilon$$

$$z-z_0 = \epsilon e^{i\theta}$$

$$\Rightarrow z = z_0+\epsilon e^{i\theta}$$

$$\Rightarrow dz = \epsilon e^{i\theta}\, i\,d\theta$$

∴ The above expression becomes –

$$\oint_C \frac{f(z)\,dz}{z-z_0} = \int_0^{2\pi} \frac{f(z_0+\epsilon e^{i\theta})\,\epsilon e^{i\theta}\, i\,d\theta}{\epsilon e^{i\theta}}$$

$$= i\int_0^{2\pi} f(z_0+\epsilon e^{i\theta})\,d\theta$$

In the limit of $\epsilon \to 0$, we get

$$\oint_C \frac{f(z)\,dz}{z-z_0} = i \lim_{\epsilon\to 0}\int_0^{2\pi} f(z_0+\epsilon e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} f(z_0)\,d\theta$$

$$= 2\pi i\, f(z_0)$$

$$\therefore f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-z_0}$$

The Cauchy Integral Formula is extended to the form –

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}},$$

where $f^{(n)}(z_0)$ denotes the n^{th} derivative of the func^n $f(z)$ at z_0.

Derivatives :-

We have, $f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)\,dz}{z - z_0}$

By defn, $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{z - z_0 - \Delta z} - \frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{z - z_0}\right]$$

or $$f'(z_0) = \frac{1}{2\pi i} \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\oint_c \left\{\frac{1}{z - z_0 - \Delta z} - \frac{1}{z - z_0}\right\} f(z)\,dz\right]$$

$$= \frac{1}{2\pi i} \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\oint_c \frac{(z - z_0 - z + z_0 + \Delta z)}{(z - z_0 - \Delta z)(z - z_0)} f(z)\,dz\right]$$

$$= \frac{1}{2\pi i} \oint_c \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left\{\frac{\Delta z}{(z - z_0 - \Delta z)(z - z_0)}\right\} f(z)\,dz$$

$$= \frac{1}{2\pi i} \oint_c \frac{f(z)\,dz}{(z - z_0)^2}$$

We assume that $f^n(z_0) = \frac{n!}{2\pi i} \oint_c \frac{f(z)\,dz}{(z - z_0)^{n+1}}$ holds true for some arbitrary integer $n = k$

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \oint_c \frac{f(z)\,dz}{(z - z_0)^{k+1}}$$

$\therefore$ $(k+1)$th derivative is given by -

$$f^{(k+1)}(z_0) = \lim_{\Delta z \to 0} \frac{f^{(k)}(z_0 + \Delta z) - f^{(k)}(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\frac{k!}{2\pi i}\oint_c \frac{f(z)\,dz}{(z - z_0 - \Delta z)^{k+1}} - \frac{k!}{2\pi i}\oint_c \frac{f(z)\,dz}{(z - z_0)^{k+1}}\right]$$

$$= \frac{k!}{2\pi i}\lim_{\Delta z \to 0} \frac{1}{\Delta z} \oint_c \left\{\frac{1}{(z - z_0 - \Delta z)^{k+1}} - \frac{1}{(z - z_0)^{k+1}}\right\} f(z)\,dz$$

$$= \frac{k!}{2\pi i}\lim_{\Delta z \to 0} \frac{1}{\Delta z} \oint_c \left\{\frac{(z - z_0)^{k+1} - (z - z_0 - \Delta z)^{k+1}}{(z - z_0 - \Delta z)^{k+1}(z - z_0)^{k+1}}\right\} f(z)\,dz$$

$$= \frac{k!}{2\pi i}\lim_{\Delta z \to 0} \frac{1}{\Delta z} \oint_c \frac{(z - z_0)^{k+1} - \{(z - z_0)^{k+1} - (k+1)(z - z_0)^k \Delta z + \cdots}{(z - z_0 - \Delta z)^{k+1}(z - z_0)^{k+1}} f(z)\,dz$$

$$= \frac{k!}{2\pi i} \oint_c \lim_{\Delta z \to 0} \frac{1}{\Delta z} \frac{(k+1)(z - z_0)^k \Delta z - O(\Delta z)^2}{(z - z_0 - \Delta z)^{k+1}(z - z_0)^{k+1}} f(z)\,dz$$

$$= \frac{(k+1)!}{2\pi i} \oint_c \frac{(z-z_0)^k}{(z-z_0)^{2k+2}} f(z)dz$$

$$= \frac{(k+1)!}{2\pi i} \oint_c \frac{f(z)dz}{(z-z_0)^{k+2}}$$

$$\therefore f^{(k+1)}(z_0) = \frac{(k+1)!}{2\pi i} \oint_c \frac{f(z)dz}{(z-z_0)^{k+2}}$$

Since, the relation holds for $n=k+1$, if it holds for $n=k$ and we have already shown that it holds for $n=1$, hence the formula holds for any +ve no.

Evaluate the ~~formula~~ following integrals.

1. $\oint_c \frac{\sin 2z}{6z-\pi} dz$ where c is the circle $|z|=3$

2. $\oint_c \frac{e^{3z}dz}{z-\ln 2}$, where c is the square with vertices $\pm 1 \pm i$

3. $\oint \frac{\sin 2z}{(6z-\pi)^3}$, where c is the circle $|z|=3$

4. $\oint_c \frac{\sin z dz}{2z-\pi}$, where c is the circle $|z|=1$

1. $\oint_c \frac{\sin 2z}{6z-\pi} dz$

$$= \frac{1}{6} \oint_c \frac{\sin 2x}{z-\frac{\pi}{6}} dz$$

$$= \frac{1}{6} \cdot 2\pi i \sin 2 \cdot \frac{\pi}{6}$$

$$= \frac{1}{6} \cdot 2\pi i \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\pi i}{2\sqrt{3}}$$

$\log_{10}{}^{10}$

2. $\oint_c \frac{e^{3z}dz}{z-\ln 2}$

$$= 2\pi i \, e^{3\ln 2}$$

$$= 2\pi i \, e^{\ln 8}$$

$$= 16\pi i$$

④ $\oint_C \frac{\sin z\,dz}{2z-\pi}$

$= \frac{1}{2} \oint_C \frac{\sin z\,dz}{\pi - \frac{\pi}{2}} = 0$

$= \frac{1}{2} . 2\pi i \sin \frac{\pi}{2}$

$= \pi i$ //

⑤ $\oint \frac{\sin 2z}{(6z-\pi)^3}$

$= \oint \frac{\sin 2z}{(6z-\pi)^{2+1}}$

$= \frac{1}{216} \oint \frac{\sin 2z}{(z-\frac{\pi}{6})^{2+1}}$

$= \frac{}{432} \oint \frac{\sin 2z}{(z-\frac{\pi}{6})^{2+1}}$

$= \frac{}{432} . 2\pi i f''(z_0)$

$= \frac{1}{432} \times 2\pi i (-4) \sin 2 . \frac{\pi}{6}$

$= -\frac{24\pi i}{108} \frac{\sqrt{3}}{2}$

$= -\frac{\sqrt{3}\pi i}{54}$

$2\cos 2z$
$-4 \sin 2z$

$\oint \frac{\sin 2z}{(6z-\pi)^3}$

$= \frac{1}{216} \oint \frac{\sin 2z}{(z-\frac{\pi}{6})^{2+1}}$

$= \frac{1}{432} \oint \frac{\sin 2z}{(z-\frac{\pi}{6})^{2+1}}$

$= \frac{1}{432} 2\pi i f''(\pi/6)$

$= \frac{1}{216} \pi i (-4) \sin 2 . \pi/6$

$= \frac{1}{216} \pi i (-4) \frac{\sqrt{3}}{2}$

$= -\frac{\sqrt{3}\pi i}{108}$ //

④ $\oint_c \frac{\cosh z\, dz}{(2\ln 2 - z)^5}$ where c is the circle $|z| = 2$

$\Rightarrow \oint \frac{\cosh z\, dz}{(2\ln 2 - z)^5}$

$\oint \frac{\cosh z\, dz}{-1(z - 2\ln 2)^5}$

$f(z) = \cosh z$

$\frac{2\pi i}{k!}$

$= -\oint \frac{\cosh z\, dz}{(z - 2\ln 2)^{4+1}}$

~~$= -\frac{2\pi i}{4!} f^4(2\ln 2)$~~

$= -\frac{2\pi i}{4!} f^4(\cosh z)$

$= -\cosh z \times \frac{2\pi i}{24\ 12}$

$= -\frac{i\pi \cosh(2\ln 2)}{12}$

$= -i\pi \left\{ \frac{e^{2\ln 2} + e^{-2\ln 2}}{24} \right\}$

$= -i\pi \left\{ \frac{4 + \frac{1}{4}}{24} \right\}$

$= -i\pi \left\{ \frac{\frac{17}{4}}{24} \right\}$

$= -i\pi \cdot \frac{17}{4} \times \frac{1}{24}$

$= -\frac{17 i\pi}{96}$

$\frac{e^z + e^{-z}}{2}$

$e^{\ln e^2}$

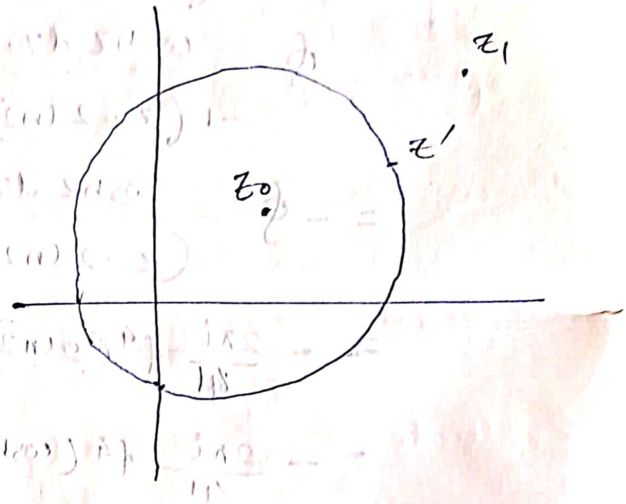
2^{-2}

$4 + \frac{1}{4}$

Taylor Expansion :-

Consider the expansion of a func^n $f(z)$ about $z = z_0$ where $z = z_1$ is the nearest pt. for which $f(z)$ is not analytic

Construct a circular contour c with centre at z_0 and radius $|z' - z_0| < |z_1 - z_0|$



Because z_1 is assumed to be the nearest pt at which $f(z)$ is non analytic. $f(z)$ is necessarily analytic on and inside c. Then by Cauchy's integral formula, ~~the value of~~

$$f(z) = \frac{1}{2\pi i} \oint_c \frac{f(z')\,dz'}{z'-z} \quad \text{--- (1)}$$

Here z' is a pt. on the contour c and z is a pt interior to c.

(1) can be re-written as,

$$f(z) = \frac{1}{2\pi i} \oint_c \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)}$$

$$\text{or } f(z) = \frac{1}{2\pi i} \oint_c \frac{f(z')\,dz'}{(z'-z_0)\left[1 - \frac{z-z_0}{z'-z_0}\right]} \quad \text{---- (2)}$$

We note the identity $\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{n=0}^{\infty} t^n$

This identity can be verified by multiplying both side by $1-t$ and this series is convergent provided $|t| < 1$.

For a pt. z interior to c $\quad |z - z_0| < |z' - z_0|$

Therefore using the identity in (2) we get,

$$f(z) = \frac{1}{2\pi i} \oint_c \frac{f(z')\,dz'}{z'-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$= \frac{1}{2\pi i} \oint_c \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(z'-z_0)^{n+1}} f(z')\,dz'$$

Interchanging the order of the integration and summation which can be done because the series is uniformly convergent.

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} \frac{n!}{2\pi i} \oint \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

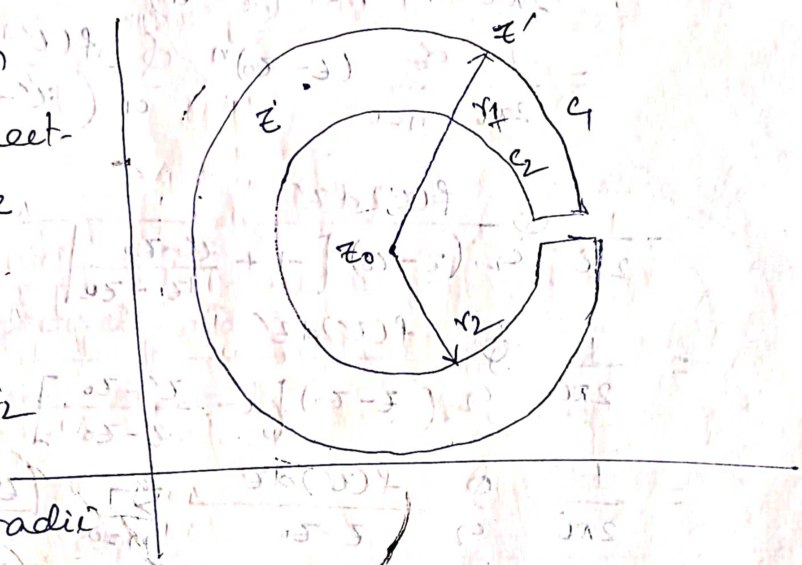
∴ we can finally write ....

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^n(z_0)}{n!}$$

which is the desired Taylor's expansion.

Laurent's series

We frequently encounter func^n s which are analytic in an annular region say of inner radius r_2 and outer radius r_1. By means of cross-cuts we can convert the annular region into a simply connected region and we apply Cauchy's integral formula for the two circles c_2 & c_1 centred at $z = z_0$ and with radii r_2 & r_1 respectively where $r_2 < r_1$



$$f(z) = \frac{1}{2\pi i} \oint_{c_1} \frac{f(z')\,dz'}{z'-z} - \frac{1}{2\pi i} \oint_{c_2} \frac{f(z')\,dz'}{z'-z} \quad \text{--- (1)}$$

Note that an explicit -ve sign has been introduced so that the contour c_2 (like c_1) is to be traversed in the +ve sense.

for c_1, $|z'-z_0| > |z-z_0|$

for c_2, $|z'-z_0| < |z-z_0|$

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')\,dz'}{z'-z} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{z'-z}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')\,dz'}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{(z-z_0)\left[-1+\frac{z'-z_0}{z-z_0}\right]}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')\,dz'}{\ldots}$$

Now

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(z')\,dz'}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]}$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')\,dz'}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')\,dz'}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$= \frac{1}{2\pi i}\oint_{C_1}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}}f(z')\,dz'$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{(z-z_0)\left[-1+\frac{z'-z_0}{z-z_0}\right]}$$

$$= \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{(z-z_0)\left[1-\frac{z'-z_0}{z-z_0}\right]}$$

$$= \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^n$$

$$= \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{z-z_0}\sum_{n=1}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^{n-1}$$

$$= \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{z-z_0}\sum_{n=1}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^{n-1}$$

$$= \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\oint_{C_2}(z'-z_0)^{n-1}f(z')\,dz'$$

$$\therefore\ f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\oint_{C_2}(z'-z_0)^{n-1}f(z')\,dz'$$

We call the 1st series S_1 and the 2nd S_2 so that

$$S_1 = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$S_2 = \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\oint_{C_2}(z'-z_0)^{n-1} f(z')\,dz'$$

We note that S_1 is the regular Taylor's expansion convergent for $|z-z_0| < |z'-z_0|$ and the 2nd series is convergent for $|z-z_0| > |z'-z_0|$.

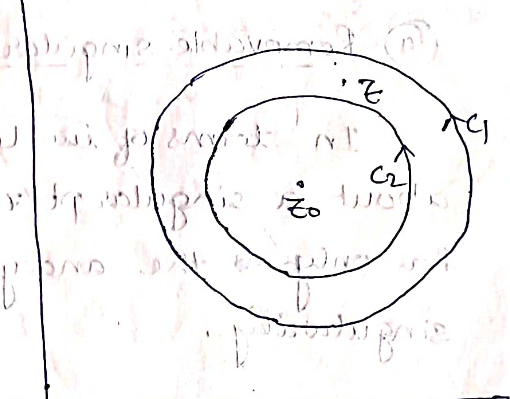
Both the series can be combined into a single series given by,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$

The above series is called the Laurent's series.

Laurent's thm :-



Let C_1 & C_2 be concentric circles of radii r_1 & r_2 respectively and let there common centre be at z_0. Suppose that, $f(z)$ is analytic and single valued on C_1 & C_2 and also in the annular region R betn them and z is any pt in R. Then,

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots + \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-3}}{(z-z_0)^3} + \ldots$$

where $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^{n+1}}, \quad n = 0, 1, 2, \ldots$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2}(z'-z_0)^{n-1} f(z')\,dz'$$

$, n = 1, 2, 3, \ldots$

C_1 & C_2 being traversed in the +ve direcⁿ w.r.t their ~~integers~~ interiors.

The circles C_1 & C_2 can be replaced by any concentric circle C betⁿ C_1 & C_2. Then the co-efficients of the above series can be obtained from a single formula

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}} \quad , \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

This is called Laurent's thm and the series above is called the Laurent's series.

The part $a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ is called the analytic part of the Laurent's series while the remainder of the series in inverse powers of $(z-z_0)$ is called the principal part. If the principal part vanishes, the Laurent's series reduces to the Taylor series.

Classification of singularities:-

ⓐ Removable singularity:-

In terms of the Laurent's expansion of a funcⁿ $f(z)$ about a singular pt z_0, we find that if this series has only the analytic part, then $z = z_0$ is a removable singularity.

ⓑ Pole:- If the Laurent's expansion of $f(z)$ about $z = z_0$ is such that, the principal part has only a finite no. of terms given by -

$$\frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-n}}{(z-z_0)^n} \quad (\text{where } a_{-n} \neq 0)$$

then z_0 is a pole of order n. If $n=1$, it is called a simple pole.

(c) Essential singularity : If $z = z_0$ is an essential singularity of $f(z)$, then the principal part of the Laurent's expansion of $f(z)$ about z_0 will have an infinite no of term

Q. Find the Laurent's series about the indicated singularity for each of the following funcⁿs.

Name the singularity in each case.

⇒ (a) $\frac{e^{2z}}{(z-1)^3}$ about $z=1$

Let $z-1 = u \Rightarrow z = u+1$

$$\frac{e^{2(u+1)}}{u^3} = \frac{e^2 \cdot e^{2u}}{u^3}$$

$$= \frac{e^2}{u^3}\left[1 + 2u + \frac{(2u)^2}{\underline{|2}} + \frac{(2u)^3}{\underline{|3}} + \frac{(2u)^4}{\underline{|4}} + \frac{(2u)^5}{\underline{|5}} + \cdots\right]$$

$$= e^2\left[\frac{1}{u^3} + \frac{2}{u^2} + \frac{2}{u} + \frac{4}{3} + \frac{2}{3}u + \frac{4}{15}u^2 + \cdots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{z-1} + \frac{4}{3} + \frac{2}{3}(z-1) + \frac{4}{15}(z-1)^2 + \cdots\right]$$

$z=1$ is a pole of order 3.

(b) $(z-3)\sin\frac{1}{z+2}$, $z=-2$

$z+2 = u$

$\Rightarrow z = u-2$

$z-3 = u-5$

$\underline{|7} = 7\times6\times5\times\underline{|4} = 5040$

$$(u-5)\sin\left(\frac{1}{u}\right) = (u-5)\left[\frac{1}{u} - \frac{\left(\frac{1}{u}\right)^3}{\underline{|3}} + \frac{\left(\frac{1}{u}\right)^5}{\underline{|5}} - \frac{\left(\frac{1}{u}\right)^7}{\underline{|7}} + \cdots\right]$$

$$= \left[\frac{u-5}{u} - \frac{u-5}{6u^3} + \frac{u-5}{u^5 \cdot 120} - \frac{u-5}{u^7\, 5040} + \cdots\right]$$

$$= \left[1 - \frac{5}{u} - \frac{1}{6u^2} + \frac{5}{6u^3} + \frac{1}{120u^4} - \frac{5}{120u^5} - \frac{1}{5040u^6} + \frac{5}{5040u^7} + \cdots\right]$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \frac{5}{120(z+2)^5} \cdots]$$

$$\frac{1}{z+2} = u$$

$$\Rightarrow z+2 = \frac{1}{u}$$

$$\Rightarrow z = \frac{1}{u} - 2$$

$$z-3 = \frac{1}{u} - 5$$

$$\left(\frac{1}{u} - 5\right) \sin u = \left(\frac{1}{u} - 5\right)\left[u - \right.$$

$\therefore$ $z = -2$ is an essential singularity.

$$\frac{1}{(z-a)^2}\left[1 + (z-a) + \frac{(z-a)^2}{2!} + \frac{(z-a)^3}{3!} + \cdots\right]$$

$$\left[\frac{1}{(z-a)^2} + \frac{1}{(z-a)} + \frac{1}{2!} + \frac{(z-a)}{} \right]$$

(c) $\frac{z - \sin z}{z^3}$

$= \frac{z - \{ z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \frac{z^7}{\lfloor 7} + \cdots \}}{z^3}$

$= \frac{1}{z^3}\left[\frac{z^3}{\lfloor 3} - \frac{z^5}{\lfloor 5} + \frac{z^7}{\lfloor 7} \cdots \right]$

$= \left[\frac{1}{\lfloor 3} - \frac{z^2}{\lfloor 5} + \frac{z^4}{\lfloor 7} - \cdots \right]$

Removable singularity.

Residue :-

Let $f(z)$ be analytic & single valued on & inside a circular contour c except a pt $z = z_0$ which is chosen as the centre of c. Then $f(z)$ has a Laurent's series about z_0 given by -

$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots$

where $a_n = \frac{1}{2\pi i} \oint_c \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \cdots$

Thus $f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$

Integrating the above series about c

$\oint_c f(z)dz = \oint_c a_0 dz + a_1 \oint_c (z-z_0) dz + a_2 \oint_c (z-z_0)^2 dz + \cdots + \cdots + a_{-1} \oint_c \frac{dz}{z-z_0} + a_{-2} \oint_c \frac{dz}{(z-z_0)^2} + \cdots$

$\Rightarrow \oint_c f(z)dz = \oint_c \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n dz$

$= \sum_{n=-\infty}^{+\infty} a_n \oint_c (z-z_0)^n dz$

Consider the terms for which $n \neq -1$ & assume that the circle has radius r, then

$|z - z_0| = r$

$\Rightarrow z - z_0 = re^{i\theta}$

$\Rightarrow z = z_0 + re^{i\theta}$

$\Rightarrow dz = ire^{i\theta} d\theta$

Then,

$$a_n \oint_c (z-z_0)^n dz = a_n \int_0^{2\pi} r^n e^{in\theta} i r e^{i\theta}$$

$$= i a_n r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= 0$$

Consider the case where $n=-1$

$$\therefore a_{-1} \oint \frac{dz}{z-z_0} = a_{-1} \int_0^{2\pi} \frac{i r e^{i\theta}}{r e^{i\theta}} d\theta$$

$$= i a_{-1} \int_0^{2\pi} d\theta$$

$$= 2\pi i a_{-1}$$

Here a_{-1} is called the residue of $f(z)$ at $z=z_0$ & is ~~called~~ also written as Res $(z-z_0)$

<u>Residue of a pole:-</u>

If $f(z)$ has a pole of order m at $z=z_0$, then the residue at z_0

$$\text{Res}(z_0) = a_{-1} = \lim_{z \to z_0} \frac{1}{\lfloor m-1} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]$$

If z_0 is a simple pole, then

$$\text{Res}(z_0) = a_{-1} = \lim_{z \to z_0} \left[(z-z_0) f(z)\right]$$

Proof:- If z_0 is a pole of order m of $f(z)$, then its Laurent's series about z_0 will be -

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1 (z-z_0) + \cdots$$

Multiplying the above by $(z-z_0)^m$

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + a_0 (z-z_0)^m + a_1 (z-z_0)^{m+1} + \cdots$$

Taking the derivatives of both sides w.r.t z, $(m-1)$ times -

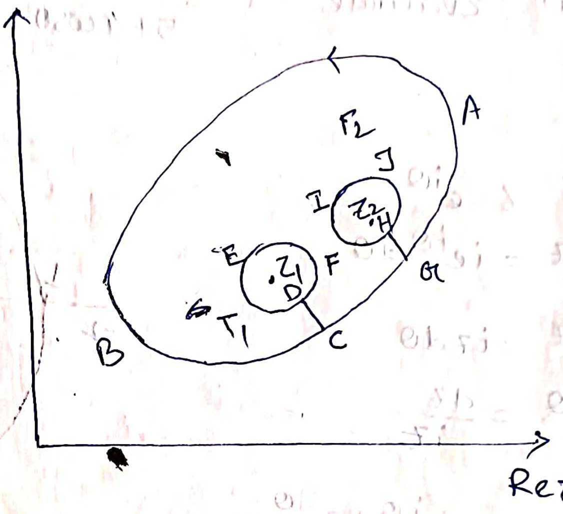
$$\frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right] = a_{-1} \lfloor m-1 + a_0 m(m-1) \cdots 2 \cdot (z-z_0) + a_1 (m+1) m \cdots (z-z_0)^2 + \cdots$$

$$\therefore \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] = a_{-1} \lfloor m-1$$

$$\therefore a_{-1} = Res(z=z_0) = \frac{1}{\lfloor m-1} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

$$a_{-1} = \lim_{z \to z_0} \frac{1}{\lfloor m-1} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

<u>Residue thm</u> If $f(z)$ is analytic on and inside a simple closed curve c except at the singularities $z_1, z_2 \ldots$ inside c then $\oint_c f(z)\,dz = 2\pi i \left[Res(z_1) + Res(z_2) + \cdots \right]$



Let c be a simple closed curve and let z_1 & z_2 be the two singular pts of the funcn $f(z)$. We encircle z_1 & z_2 by circular contours Γ_1 & Γ_2 and connect Γ_1 & Γ_2 with c as shown in the diagram. If we now consider the curve ABCDEFDCGHIJGA, then z_1 & z_2 are outside the contours and hence by Cauchy's thm we get,

$$\int_{ABCDEFDCGHIJGA} f(z)\,dz = 0$$

$$\Rightarrow \int_{ABC} + \int_{CD} + \int_{DEFD} + \int_{DC} + \int_{CG} + \int_{GH} + \int_{HIJH} + \int_{HG} + \int_{GA} = 0$$

$$\Rightarrow \int_{ABC} + \int_{CG} + \int_{GA} + \int_{DEFD} + \int_{HIJH} = 0$$

$$\Rightarrow \oint_c f(z)\,dz + \oint_{\Gamma_1} f(z)\,dz + \oint_{\Gamma_2} f(z)\,dz = 0$$

$$\Rightarrow \oint_c f(z)\,dz = \oint_{\Gamma_1} f(z)\,dz + \oint_{\Gamma_2} f(z)\,dz$$

$$= 2\pi i \, Re(z_1) + 2\pi i \, Re(z_2)$$

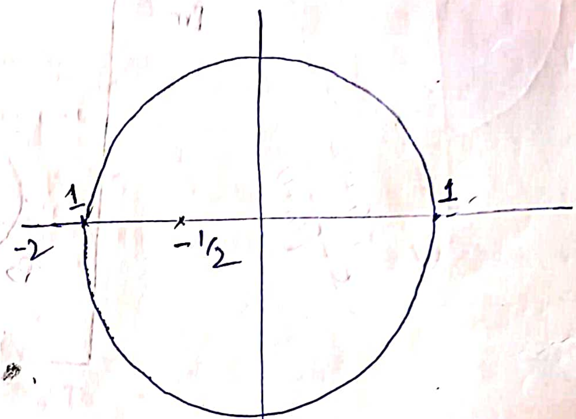
$$\oint_C f(z)\,dz = 2\pi i\left[\text{Res}(z_1) + \text{Res}(z_2)\right]$$

This proof can be extended to the case where the no. of singularities is more than two.

Evaluation of Definite Real Ingrals:-

(A) Integrals of the type $\int_0^{2\pi} f\left\{\begin{matrix}\cos\theta \\ \sin\theta\end{matrix}\right\} d\theta$

(i) Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$.



let $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$dz = iz\,d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$= \frac{z+\frac{1}{z}}{2}$

$\therefore I = \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{1}{2}\int \frac{z^2+1}{z}\,dz$

By substituting -

$I = \oint_C \frac{\frac{1}{iz}\,dz}{5+\frac{4}{2}\left(\frac{z^2+1}{z}\right)}$, where C is the unique circle with centre at the origin in complex plane.

$= \frac{1}{i}\oint_C \frac{dz}{z\left\{\frac{5z+2z^2+2}{z}\right\}}$

$= \frac{1}{i}\oint_C \frac{dz}{2z^2+5z+2}$

$= \frac{1}{i}\oint_C \frac{dz}{2z^2+4z+z+2}$

$= \frac{1}{i}\oint \frac{dz}{(2z+1)(z+2)}$

$$= \frac{1}{2i} \oint \frac{dz}{(z+\frac{1}{2})(z+2)}$$

$$Res \left| z = -\frac{1}{2} \right| = \lim_{z \to -\frac{1}{2}} \left[(z+\tfrac{1}{2}) \frac{1}{(z+\frac{1}{2})(z+2)} \right]$$

$$= \frac{1}{3/2}$$

$$= \frac{2}{3}$$

$$\therefore \quad I = 2\pi i \frac{1}{2i} \times \frac{2}{3}$$

$$= \frac{2\pi}{3}$$

(ii) $$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$$

$z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$dz = iz\, d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{i}\left[\frac{z - \frac{1}{z}}{2}\right] = \frac{1}{2i}\left(\frac{z^2-1}{z}\right)$

$$\frac{d\theta}{13+5\sin\theta} = \frac{\frac{1}{iz}\, dz}{13 + 5 \cdot \frac{1}{2i}\left(\frac{z^2-1}{z}\right)}$$

$$= \frac{\frac{1}{iz}\, dz}{\frac{26iz + 5z^2 - 5}{2iz}}$$

$$= \frac{dz}{iz} \times \frac{2iz}{26iz + 5z^2 - 5}$$

$$= \frac{2\, dz}{26iz + 5z^2 - 5}$$

$$= \frac{2\, dz}{26iz + 5z^2 + 5(i)^2}$$

$$= \frac{2\, dz}{5z^2 + 25iz + iz + 5(i)^2}$$

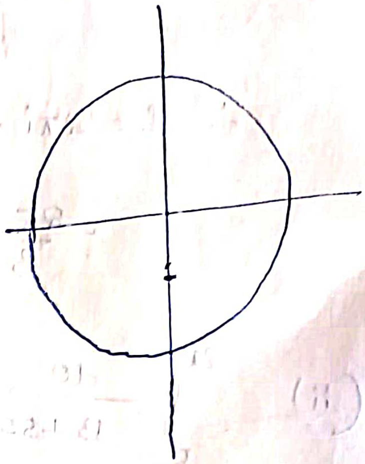
$$= \frac{2\, dz}{5z(z+5i) + i(z+5i)}$$

$$= \frac{2\, dz}{(z+5i)(5z+i)}$$

$$= \frac{2}{5} \frac{dz}{(z+5i)(z+i/5)}$$

$$I = \frac{2}{5}\oint \frac{dz}{(z+5i)(z+i/5)}$$

$$Res|_{z=-i/5} = \lim_{z\to -i/5} (z+i/5)\frac{1}{(z+5i)(z+i/5)}$$



$$= \frac{1}{-\frac{1}{5}+5i}$$

$$= \frac{}{\frac{-i+25i}{5}}$$

$$= \frac{5}{25i-i}$$

$$= \frac{5}{24i}$$

$$\therefore I = 2\pi i \frac{5}{24i}\times\frac{2}{5}$$

$$= \frac{\pi}{6}$$

Evaluate the following integral :-

1. $$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta}$$

$z = e^{i\theta}$

$\Rightarrow dz = ie^{i\theta}d\theta$

$\Rightarrow dz = iz\,d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

$(e^{i\theta})^2+(e^{-i\theta})^2$

$z^2-\frac{1}{z^2}$

$$\frac{\cos 2\theta}{5+4\cos\theta}$$

$$= \frac{e^{i2\theta}+e^{-i2\theta}}{2\{5+4\cdot\frac{e^{i\theta}+e^{-i\theta}}{2}\}}$$

$$= \frac{e^{i2\theta}+e^{-i2\theta}}{10+4(e^{i\theta}+e^{-i\theta})}$$

$$= \frac{z^2+\frac{1}{z^2}}{10+4(z+\frac{1}{z})}$$

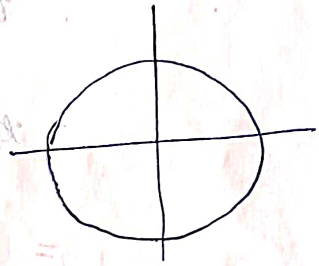
$$= \frac{z^4+1}{z^2\left\{10+4\left(\frac{z^2+1}{z}\right)\right\}}$$

$$= \frac{z^4-1}{z\,z^2\left\{\frac{10z+4z^2+4}{z}\right\}}$$

$$= \frac{z^4+1}{4z^3+10z^2+4z}$$

$$= \frac{z^4+1}{z\left(4z^2+10z+4\right)}$$

$$= \frac{z^4+1}{z\left(4z^2+8z+2z+4\right)}$$



$$= \frac{z^4+1}{z\left\{4z(z+2)+2(z+2)\right\}}$$

$$= \frac{z^4+1}{z\,(4z+2)\,(z+2)}$$

$$\oint \frac{(z^4+1)\,dz}{4z\,(z+\frac{1}{2})(z+2)\;iz}$$

$$= \frac{1}{4i}\oint \frac{(z^4+1)\,dz}{z^2\,(z+\frac{1}{2})(z+2)}$$

$$Res\Big|_{z=-\frac{1}{2}} = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\frac{(z^4+1)}{z^2\,(z+\frac{1}{2})\,(z+2)}$$

$$= \frac{z^4+1}{z^2\,(z+2)}$$

$$= \frac{\frac{1}{16}+1}{\frac{1}{4}\left(2-\frac{1}{2}\right)}$$

$$= \frac{17/16}{\frac{1}{4}\cdot\frac{3}{2}}$$

$$= \frac{17/16}{3/8}$$

$$= \frac{17}{16}\times\frac{8}{3}$$

$$= \frac{17}{6}$$

$$2\pi i \times \frac{17}{6}\times\frac{1}{4i} = \frac{17\pi}{12}$$

$$\text{Res}\Big|_{z=0} = \lim_{z\to 0} \frac{d}{dz}\left[z^2 \frac{(z^4+1)}{z^2 (z+\frac{1}{2})(z+2)} \right]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left[\frac{z^4+1}{(z+\frac{1}{2})(z+2)} \right]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left[\frac{(z^4+1)}{z^2+2z+\frac{z}{2}+1} \right]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left[\frac{(z^4+1)}{z^2+\frac{5}{2}z+1} \right]$$

$$= \lim_{z\to 0} \frac{(z^2+\frac{5}{2}z+1)(4z^3) - (z^4+1)(2z+\frac{5}{2})}{(z^2+\frac{5}{2}z+1)^2}$$

$3z^3$

$9z^2$

$18z$

18

$2\pi i \times \frac{1}{4i} \times 18$

$$= \frac{-1.\ 5/2}{(1)}$$

$$= -\frac{5}{2}$$

$$\left(\frac{17}{6} - \frac{5}{2}\right) = \frac{17-15}{6} = \frac{2}{6} = +\frac{1}{3}$$

$$2\pi i \times \frac{1}{4i} \times \frac{1}{3}$$

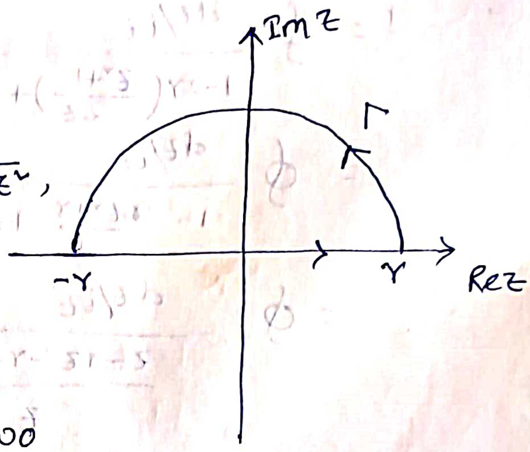
$$= \frac{\pi}{6}$$

B) Integrals of the form $\int_{-\infty}^{\infty} F(x)\,dx$

Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Consider the complex integral $I = \oint_c \frac{dz}{1+z^2}$ where c is the contour shown in the diagram.

We split up the integral I as follows -



$$\oint_c \frac{dz}{1+z^2} = \int_{-r}^{r} \frac{dx}{1+x^2} + \int_{\Gamma} \frac{dz}{1+z^2},$$

where r is the radius of the semi-circle contour.

The integrand in I has two simple poles $z = \pm i$

$z = -i$ is outside the contour.

$$\text{Res}\,(z=i) = \lim_{z\to i} (z-i) \frac{1}{(z+i)(z-i)}$$

$$= \frac{1}{2i}$$

$$I = 2\pi i \,\text{Res}(z=i) = 2\pi i \times \frac{1}{2i} = \pi$$

In the integral over Γ, $z = re^{i\theta}$

$$dz = rie^{i\theta}d\theta$$

$$\int_{\Gamma} \frac{dz}{1+z^2} = \int_{0}^{\pi} \frac{rie^{i\theta}\,d\theta}{1+r^2e^{2i\theta}}$$

$$\therefore \pi = \int_{-r}^{r} \frac{dx}{1+x^2} + \int_{\theta=0}^{\pi} \frac{rie^{i\theta}\,d\theta}{1+r^2e^{i2\theta}}$$

Taking the limit of the above as $r \to \infty$, we get,

$$\lim_{r\to\infty} \int_{-r}^{r} \frac{dx}{1+x^2} + \lim_{r\to\infty} \int_{\theta=0}^{\pi} \frac{rie^{i\theta}\,d\theta}{1+r^2e^{i2\theta}}$$

The 2nd integral vanishes as $r \to \infty$ and therefore

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

2) $\int_0^\infty \frac{x^2 dx}{x^4+16}$

$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4+16}$

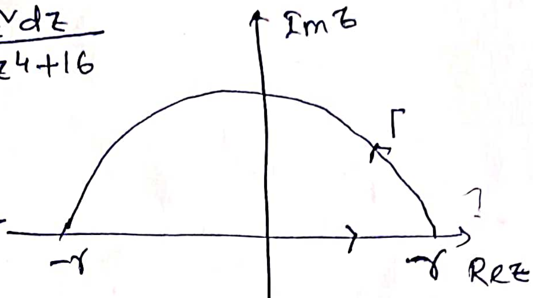
consider the complex integral $I = \int \frac{z^2 dz}{z^4+16}$

$(z^2)^2 - (4i)^2$

$(z^2+4i)(z^2-4i)$

$z^2=-4i \quad z^2=4i$

$\oint \frac{z^2 dz}{z^4+16} = \int_{-r}^{r} \frac{x^2 dx}{x^4+16} + \int_\Gamma \frac{z^2 dz}{z^4+16}$



$\text{Res}(z=2\sqrt{i}) = \lim_{z\to 2\sqrt{i}} (z-2\sqrt{i}) \frac{z^2}{(z-2\sqrt{i})(z+2\sqrt{i})(z-2\sqrt{i^3})(z+2\sqrt{i^3})}$

$= \frac{z^2}{(z+2\sqrt{i})(z-2i\sqrt{i})(z+2i\sqrt{i})}$

$= \frac{(2\sqrt{i})^2}{(2\sqrt{i}+2\sqrt{i})(2\sqrt{i}-2i\sqrt{i})(2\sqrt{i}+2i\sqrt{i})}$

$= \frac{4i}{4\sqrt{i}\; 2\sqrt{i}(1-i)\; 2\sqrt{i}(1+i)}$

$= \frac{4i}{4 \cdot 16\, i \sqrt{i}\,(1-i^2)}$

$= \frac{1}{4\sqrt{i}\, 2}$

$= \frac{1}{8\sqrt{i}}$

$z^2 = -4i$

$z^2 = 4i$

$z = \pm 2\sqrt{i}$

$z^2 = 4i^3$

$z = \pm 2\sqrt{i^3}$

$(z-2\sqrt{i})(z+2\sqrt{i})(z-2\sqrt{i^3})(z+2\sqrt{i^3})$

$2\sqrt{i}(1-i)$

H/W

$\int_{-\infty}^{\infty} \frac{dx}{(4x^2+1)^3}$

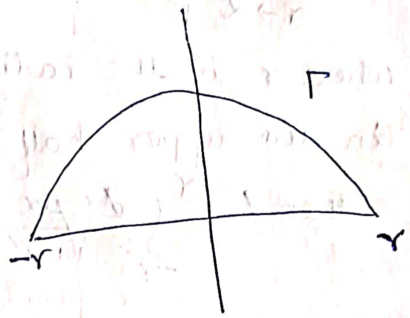
3)

⇒ Integrals of the type $\int_{-\infty}^{\infty} f(x) \left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\} dx$

Evaluate $\int_0^{\infty} \frac{\cos x \, dx}{1+x^2}$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x \, dx}{1+x^2}$$

Consider the integral.



$$I = \oint \frac{e^{iz} \, dz}{1+z^2}$$

$$= \oint \frac{e^{iz}}{(z+i)(z-i)}$$

$$\text{Res}(z=i) = \lim_{z \to i} (z-i) \frac{e^{iz}}{(z+i)(z-i)}$$

$$= \frac{e^{-1}}{2i}$$

$$\therefore \quad I = 2\pi i \ \text{Res}(i)$$

$$= 2\pi i \times \frac{e^{-1}}{2i}$$

$$= \frac{\pi}{e}$$

$$\oint \frac{e^{iz} dz}{1+z^2} = \int_{-r}^{r} \frac{e^{ix} dx}{1+x^2} + \int_{\Gamma} \frac{e^{iz} dz}{1+z^2}$$

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix} \cdot e^{-y}| = |e^{ix}| \, |e^{-y}|$$

$$= e^{-y} \leq 1$$

The 2nd part of the integral vanishes as $r \to \infty$

$$\therefore \quad \frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{\cos x \, dx}{1+x^2} + i \int_{-\infty}^{\infty} \frac{\sin x \, dx}{1+x^2}$$

equating the real and imaginary parts -

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{1+x^2} = \frac{\pi}{e},$$

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{1+x^2} = 0$$

Jordan's formula :-

If $F(z)$ is analytic in the upper half plane and $|F(z)| \to 0$ as $|z| \to 0$ as $|z| \to \alpha$ then -

$$\lim_{r \to \alpha} \int_{\Gamma} e^{i\alpha z} F(z)\, dz = 0 \quad , \quad \alpha > 0$$

where r is the radius of the semi circular contour Γ in the upper half plane.

~~then~~ $\int_{-r}^{r} \frac{dx}{1+x^2} + \oint \frac{e^{iz}\, dz}{1+z^2}$

Evaluate -

① $\int_{0}^{\alpha} \frac{\cos x \, dx}{9x^2+4}$

$$= +\frac{1}{2} \int_{-\alpha}^{\alpha} \frac{\cos x \, dx}{9x^2+4}$$

$$= +\frac{1}{18} \int_{-\alpha}^{\alpha} \frac{\cos x \, dx}{x^2+\frac{4}{9}}$$

Let $I = \oint \frac{e^{iz}\, dz}{z^2+\frac{4}{9}}$

$$I = \oint \frac{e^{iz}\, dz}{(z+\frac{2i}{3})(z-\frac{2i}{3})}$$

$$\text{Res}\left(z=\frac{2i}{3}\right) = \lim_{z \to \frac{2i}{3}} \left(z-\frac{2i}{3}\right) \frac{e^{iz}}{(z+\frac{2i}{3})(z-\frac{2i}{3})}$$

$$= \frac{e^{i \cdot 2i/3}}{4i/3}$$

$$= \frac{3e^{-2/3}}{4i}$$

$$I = 2\pi i \, \text{Res}(z = 2i/3)$$

$$= 2\pi i \times \frac{3e^{-2/3}}{4i}$$

$$= \frac{3}{2} \pi e^{-2/3}$$

$$\therefore I = \int_{-r}^{r} \frac{\cos x \, dx}{x^2+4/9} + \int_{\Gamma} \frac{e^{iz}}{z^2+\frac{4}{9}} dz$$

By Jordan ~~formula~~ Lemma. The 2nd integral -

$$\therefore \frac{3\pi}{2e^{2/3}} = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4/9}\,dx$$

$$\Rightarrow \frac{3\pi}{2e^{2/3}} = \int_{-\infty}^{\infty} \frac{\cos x\,dx}{x^2+4/9} + i\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+4/9}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x\,dx}{x^2+4/9} = \frac{3\pi}{2e^{2/3}}$$

$$\therefore \int_{0}^{\infty} \frac{\cos x\,dx}{9x^2+4} = \frac{1}{\cancel{18}\,6} \times \frac{\cancel{3}\pi}{2e^{2/3}}$$

$$= \frac{\pi}{12e^{2/3}}$$