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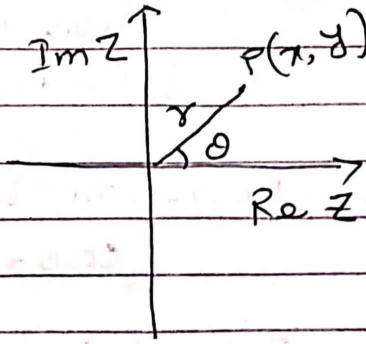
Polar form of complex No. :-

$$z = r(\cos\theta + i\sin\theta)$$
$$r = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$



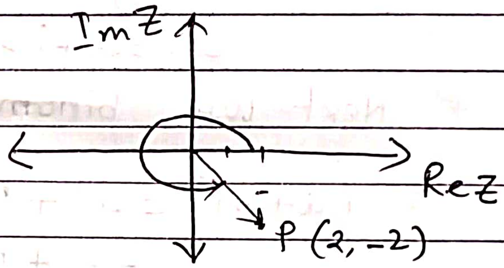
Ex: (i) $2-2i$

Here $x = 2$; $y = -2$

$$r = \sqrt{x^2+y^2} = \sqrt{2^2+(-2)^2} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(-\frac{2}{2}\right) = \tan^{-1}(-1)$$

Now check where z lies in the complex plane. The no. will lie in the 4th quadrant as shown in the Fig.



Again $\tan \pi/4 = 1$

So, $\theta = \tan^{-1}(-1)$

$$= \tan^{-1}\{\tan(2\pi - \pi/4)\}$$
$$= 7\pi/4$$

$$\left\{ \begin{aligned} \theta &= \tan^{-1}(-1) \\ &= \tan^{-1}(\tan(2\pi - \pi/4)) \\ &= 7\pi/4 \end{aligned} \right\}$$

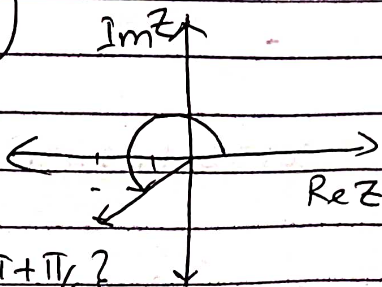
$$\therefore 2-2i = 2\sqrt{2}\left\{\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right\} \#$$

(ii) $-2\sqrt{3}-2i$

Here $x = -2\sqrt{3}$ $y = -2$

$$r = \sqrt{x^2+y^2} = \sqrt{16} = 4$$

$$\theta = \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$



The point will lie in the 3rd quadrant and $\tan \pi/6 = \frac{1}{\sqrt{3}}$

So, $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\tan\{\pi + \pi/6\}$

$$= \theta = 7\pi/6$$

$$\therefore -2\sqrt{3}-2i = 4\{\cos 7\pi/6 + i\sin 7\pi/6\}$$

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Assignment 1 : Express each of the following complex no.s in polar form
(i) $-1+\sqrt{3}i$ (ii) $2\sqrt{2}+2\sqrt{2}i$ (iii) $-i$ (iv) -4 (v) $\sqrt{2}i$

De-Moivre's thm :-
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

This is true for any positive, negative integer and also valid for rational numbers i.e $n = p/q$.

Ex. Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Now using De-Moivre's thm
$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

Next use binomial expansion of $(a+b)^n$

$$(a+b)^n = a^n + {}^nc_1 a^{n-1} b + {}^nc_2 a^{n-2} b^2 + \cdots$$
$$\cdots + {}^nc_r a^{n-r} b^r + \cdots + b^n$$

where ${}^nc_r = \dfrac{n!}{r!\,(n-r)!}$

So, $\cos 3\theta + i\sin 3\theta = (\overset{a}{\cos\theta} + \overset{b}{i\sin\theta})^3$
$$= \cos^3\theta + {}^3c_1 \cos^2\theta\,(i\sin\theta) + {}^3c_2 \cos\theta\,(i\sin\theta)^2 + {}^3c_3 (i\sin\theta)^3$$
$$= \cos^3\theta + 3\cos^2\theta\sin\theta\, i + 3\cos\theta(-\sin^2\theta) - i\sin^3\theta$$

$$\left[\begin{aligned} i^3 &= i^2 . i \\ &= -1 . i = -i \end{aligned}\right]$$

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

comparing real and imaginary parts $\rightarrow$
$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

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$$\cos 3\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$
$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta \quad \#$$

$$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$$
$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta \quad \#$$

Assignment 2: Prove that (a) $\frac{\sin 4\theta}{\sin\theta} = 8\cos^3\theta - 4\cos\theta$

(b) $\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1$

Roots of complex no.:

if $z = r(\cos\theta + i\sin\theta)$

$$z^{1/n} = \{r(\cos\theta + i\sin\theta)\}^{1/n}$$
$$= r^{1/n}(\cos\theta + i\sin\theta\}^{1/n}$$

Now $\cos(\theta + 2k\pi) = \cos\theta$ for $k = 0, 1, 2, \ldots$

$\sin(\theta + 2k\pi) = \sin\theta$ for $k = 0, 1, 2 \ldots$

So, $z^{1/n} = r^{1/n}\{\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\}^{1/n}$

where $k = 0, 1, 2 \ldots n-1$

$$\therefore z^{1/n} = r^{1/n}\left\{\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right\}$$

→ Using De Moivre's th^m

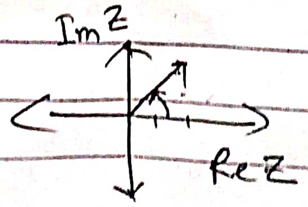
Ex. (i) $(2 + 2\sqrt{3}\,i)^{1/3}$

in polar form →

$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + (2\sqrt{3})^2} = 4$$

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}(\sqrt{3})$$

$$\theta = \tan^{-1}\tan\pi/3 = \pi/3$$



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$$\text{So, } \left(2+2\sqrt{3}\,i\right)^{1/3} = 4^{1/3}\left\{\cos\frac{\pi/3+2k\pi}{3} + i\sin\frac{\frac{\pi}{3}+2k\pi}{3}\right\}$$

where $k=0,1,2$

$$1^{st}\text{ root} = 4^{1/3}\left\{\cos\frac{\pi/3}{3} + i\sin\frac{\frac{\pi}{3}}{3}\right\}, \text{ for } k=0$$

$$= 4^{1/3}\left\{\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right\}$$

$$2^{nd}\text{ root} = 4^{1/3}\left\{\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right\}, \text{ for } k=1$$

$$3^{rd}\text{ root} = 4^{1/3}\left\{\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right\}, \text{ for } k=2$$

#

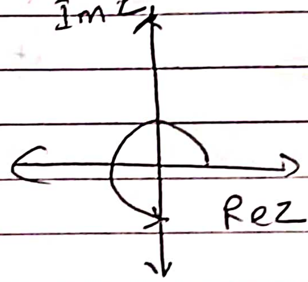
(ii) $(-16i)^{1/4} = \left\{0+(-16)i\right\}^{1/4}$

$$r = \sqrt{0^2+(-16)^2} = 16$$

$$\theta = \tan^{-1}\left(-\frac{16}{0}\right) = \tan^{-1}(\alpha)$$

$$= \tan^{-1}\left[\tan\left(\pi+\pi/2\right)\right]$$

$$= 3\pi/2$$



$$\text{So, } (-16i)^{1/4} = (16)^{1/4}\left\{\cos\frac{3\pi/2+2k\pi}{4} + i\sin\frac{3\pi/2+2k\pi}{4}\right\}$$

$k=0,1,2,3$

$$1^{st}\text{ root} = 2\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right), \quad k=0$$

$$2^{nd}\text{ root} = 2\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right), \quad k=1$$

$$3^{rd}\text{ root} = 2\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right), \quad k=2$$

$$4^{th}\text{ root} = 2\left(\cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}\right), \quad k=3$$

Assignment : 3 Find roots

(i) $\left(2\sqrt{3}-2i\right)^{1/2}$ (ii) $(64)^{1/6}$ (iii) $(-1)^{1/3}$