Working Rule for solving $Pp + Qq = R$ by Lagrange's Method:

Step 1: Put the given linear PDE in the standard form

$$Pp + Qq = R \longrightarrow (1)$$

Step 2: Write down Lagrange's auxiliary equations for (1) as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \longrightarrow (2)$$

Step 3: Solve (2) by using the well known methods.

Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4: The general solution (integral) of (1) is then written in one of the following three equivalent forms:

$$\phi(u, v) = 0, \quad u = \phi(v) \quad \text{or} \quad v = \phi(u).$$

— x —

Example 1: Solve $xp + yq = 3$

Sol^n: Given that,

$$xp + yq = 3 \longrightarrow (1)$$

Comparing the given PDE (1) with $Pp + Qq = R$ we get $P = x,\ Q = y,\ R = 3$.

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3} \longrightarrow (2)$$

Taking the first two fractions of (2), we get

$$\frac{dx}{x} = \frac{dy}{y} \longrightarrow (3)$$

Integrating (3), we get

$$\log x = \log y + \log c_1$$

$$\Rightarrow x/y = c_1$$

Again, taking last two fractions of (2), we get

$$\frac{dy}{y} = \frac{dz}{3} \longrightarrow (4)$$

Integrating (4), we get

$$\log y = \log 3 + \log c_2$$

$$\Rightarrow \frac{y}{z} = c_2$$

Hence the required general integral is

$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$, where ϕ being an arbitrary function.

— x —

Example 2: Solve $(y+z)p + (z+x)q = x+y$

Soln: Given that,

$$(y+z)p + (z+x)q = x+y \longrightarrow (1)$$

where $P = y+z$, $Q = z+x$, $R = x+y$

Now, Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \longrightarrow (2)$$

choosing $1, -1, 0$; $0, 1, -1$ and $1, 1, 1$ as multipliers in turn, each fraction of (2)

$$= \frac{dx - dy}{y - x} = \frac{dy - dz}{z - y} = \frac{dx + dy + dz}{2(x+y+z)}$$

$$\therefore \frac{dx - dy}{y - x} = \frac{-(dz - dy)}{z - y} = \frac{dx + dy + dz}{2(x+y+z)} \rightarrow (3)$$

Taking the 1st two fractions of 3, we have

$$\frac{dx-dy}{y-x} = \frac{-(dz-dy)}{z-y}$$

Integrating,

$$-\log(y-x) = -\log(z-y) + \log c_1$$

$$\Rightarrow \quad \frac{z-y}{y-x} = c_1 \longrightarrow (4)$$

Again, taking last two fractions of 3, we get

$$-\frac{(dz-dy)}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\Rightarrow \quad \frac{2(dz-dy)}{z-y} = -\frac{dx+dy+dz}{x+y+z}$$

Integrating,

$$2\log(z-y) = -\log(x+y+z) + \log c_2$$

$$\Rightarrow (z-y)^2(x+y+z) = c_2 \longrightarrow (5)$$

From, 4 & 5, the required general solution is $\phi\left(\frac{z-y}{y-x}, (z-y)^2(x+y+z)\right) = 0$,

where ϕ being an arbitrary function.

—— x ——

Example 3: Solve $xzp + yzq = xy$

Sol^n: Given that, $xzp + yzq = xy \longrightarrow (1)$

Here the Lagrange's auxiliary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \longrightarrow (2)$$

From the 1^{st} two fractions of (1), we get

$$\frac{dx}{xz} = \frac{dy}{yz}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$\log x = \log y + \log c_1$$

$$\Rightarrow \frac{x}{y} = c_1$$

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1) $= \dfrac{1/x\, dx + 1/y\, dy}{\left(\frac{1}{x}\right)xz + \left(\frac{1}{y}\right)yz}$

$$= \frac{y\,dx + x\,dy}{2xyz} \longrightarrow (3)$$

Taking last fraction of (2) and (3), we get

$$\frac{dz}{xy} = \frac{y\,dx + x\,dy}{2xyz}$$

$$\Rightarrow y\,dx + x\,dy = 2z\,dz$$

$$\Rightarrow d(xy) = 2z\,dz$$

Integrating,

$$xy - z^2 = c_2$$

Hence the required general solution is

$$\phi\left(x/y,\ xy - z^2\right) = 0.$$

where ϕ being an arbitrary function.

Example 4: Solve $z(z^2+xy)(px-qy) = x^4$

Soln: Given that,

$$z(z^2+xy)(px-qy) = x^4$$

$$\Rightarrow xz(z^2+xy)p - yz(z^2+xy)q = x^4 \longrightarrow (1)$$

Here, $P = xz(z^2+xy)$

$Q = -yz(z^2+xy)$

$R = x^4$

The Lagrange's auxiliary equations for ① are

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4} \longrightarrow (2)$$

Taking the 1^{st} two fractions of ②, we get

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dy}{-y}$$

Integrating,

$$\log x = -\log y + \log c_1$$

$$\Rightarrow \quad xy = c_1 \longrightarrow (3)$$

Taking 1^{st} & 3^{rd} fractions of ②, we get

$$\frac{dx}{xz(z^2+xy)} = \frac{dz}{x^4}$$

$$\Rightarrow \quad \frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4} \qquad [\text{using } (3)]$$

$$\Rightarrow \quad \frac{dx}{z(z^2+c_1)} = \frac{dz}{x^3}$$

$$\Rightarrow \quad x^3\,dx - z^3\,dz - c_1 z\,dz = 0$$

Integrating, we get

$$\frac{x^4}{4} - \frac{z^4}{4} - c_1 \frac{z^2}{2} = c_2'$$

$$\Rightarrow x^4 - z^4 - 2c_1 z^2 = c_2 \quad [\text{Taking } c_2 = 4c_2'] \longrightarrow (4)$$

From (3) and (4), we get the general solution as

$$\phi\left(xy,\ x^4 - z^4 - 2c_1 z^2\right) = 0$$

where ϕ being an arbitrary function. //

Remark: If $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \longrightarrow (1)$

and P_1, Q_1, R_1 be functions of x, y and z, then by a well-known principle of algebra, each fraction ~~of~~ in (1) will be equal to

$$\frac{P_1\,dx + Q_1\,dy + R_1\,dz}{P_1 P + Q_1 Q + R_1 R}. \longrightarrow (2)$$

If $P_1 P + Q_1 Q + R_1 R = 0$, then the numerator of (2) is also zero.

Example 5: Solve $x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$

Sol^n: Given that,

$$x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2) \longrightarrow (1)$$

Here $P = x(y^2-z^2)$, $Q = -y(z^2+x^2)$,

$R = z(x^2+y^2)$

The Lagrange's auxiliary equations of the given equation (1) are

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} \longrightarrow (2)$$

Choosing x, y, z as multipliers, each fraction of (2)

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) - y^2(z^2+x^2) + z^2(x^2+y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow x\,dx + y\,dy + z\,dz = 0$$

Integrating, $x^2+y^2+z^2 = c_1$

Choosing $1/x, -1/y, -1/z$ as multipliers, each fraction of (2) $= \dfrac{dx/x - dy/y - dz/z}{y^2 - z^2 + z^2 + x^2 - (x^2+y^2)}$

$$= \frac{dx/x - dy/y - dz/z}{0}$$

$$\Rightarrow \frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} = 0$$

Integrating,

$$\log x - \log y - \log z = \log c_2$$

$$\Rightarrow \frac{x}{yz} = c_2$$

Hence the required general solution is

$$\phi\left(x^2+y^2+z^2, \frac{x}{yz}\right) = 0,$$

where ϕ being an arbitrary function. //

Lagrange's Method for more than Two Independent Variables :

Let z be a function of n independent variable $x_1, x_2, \dots, x_n$.

Let us consider the notations :

$$\frac{\partial z}{\partial x_i} = p_i \quad , \; i = 1, 2, \dots, n.$$

The Lagrange's equation can now written as

$$P_1 p_1 + P_2 p_2 + \dots + P_n p_n = R \longrightarrow (1)$$

where $P_1, P_2, \dots, P_n, R$ are functions of $z, x_1, x_2, \dots, x_n$.

To solve (1), we have to find n independent solutions of auxiliary equations,

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dz}{R} \longrightarrow (2)$$

If these solutions are

$$u_1 = c_1, \; u_2 = c_2, \dots, u_n = c_n,$$

then the complete solution of (1) is given by $\phi(u_1, u_2, \dots, u_n) = 0$, where ϕ being an arbitrary function. //

Example : Solve $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=az+\frac{xy}{t}$

Sol^n: Given that,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=az+\frac{xy}{t} \longrightarrow (1)$$

The Lagrange's auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{az+xy/t} \longrightarrow (2)$$

Taking 1^{st} and 2^{nd} fractions of (2), we get

$$\frac{dx}{x}=\frac{dy}{y}$$

Integrating, $\log x = \log y + \log c_1$

$$\Rightarrow x/y = c_1 \longrightarrow (3)$$

Taking 1^{st} and 3^{rd} fractions of (2), we get

$$\frac{dx}{x}=\frac{dt}{t}$$

Integrating, $\log x = \log t + \log c_2$

$$\Rightarrow x/t = c_2 \longrightarrow (4)$$

Again, taking 1st and 4th fractions of ②, we get

$$\frac{dx}{x} = \frac{dz}{az + xy/t}$$

$$\Rightarrow \frac{dz}{dx} = \frac{az + xy/t}{x}$$

$$\Rightarrow \frac{dz}{dx} - \frac{a}{x} z = \frac{c_2}{c_1} \quad [\text{using } ③ \,\&\, ④] \longrightarrow ⑤$$

Now we wish to solve ⑤.

Equation ⑤ is in the form

$$\frac{dz}{dx} + Pz = Q$$

$$\therefore \quad I.F. = e^{-\int a/x \, dx} = e^{-a \log x} = x^{-a}$$

Thus, solⁿ of ⑤ is given by

$$z\,(I.F.) = \int [Q.(I.F.)]\, dx + c_3$$

$$\Rightarrow z\, x^{-a} = \int \frac{c_2}{c_1} x^{-a}\, dx + c_3$$

$$\Rightarrow z x^{-a} = \frac{c_2}{c_1} \cdot \frac{x^{1-a}}{1-a} + c_3$$

$$\Rightarrow \frac{z}{x^a} = \frac{y}{t} \cdot \frac{x^{1-a}}{1-a} + c_3 \longrightarrow [\text{using } ③ \,\&\, ④] \longrightarrow ⑥$$

Hence the required general sol^n is given by

$$\phi\left(\frac{y}{x}, \frac{t}{x}, \frac{z}{x^a} - \frac{y}{t}\frac{x^{1-a}}{1-a}\right) = 0,$$

where ϕ being an arbitrary function. //

References:

(1) Raisinghania, M. D.(2008). Advanced Differential Equations, S. Chand & Company Ltd., India.

(2) Sharma, B. D.(2018). Differential Equations, 1st Revised Edition, Kedar Nath Ram Nath, Meerut.