Paper: C 2.1 (Real Analysis)

Unit I

Finite and Infinite Sets:

- ✓ The empty set ϕ is said to have 0 elements.
- ✓ If $n \in \mathbb{N}$ (set of natural numbers), a set S is said to have n elements if there exists a bijection from the set $\mathbb{N}_n = \{1, 2, \dots, n\}$ onto S.
- ✓ A set S is said to be **finite** if it is either empty or it has n elements for some $n \in \mathbb{N}$.
- ✓ A set S is said to be **infinite** if it is not finite.

Uniqueness Theorem: If S is a finite set, then the number of elements in S is a unique number in $\mathbb{N}$.

Theorem: The set $\mathbb{N}$ of natural numbers is an infinite set.

Proof: Let us assume that $\mathbb{N}$ is a finite set.

$\Rightarrow \exists$ a bijection $f : \mathbb{N}_n \to \mathbb{N}$ for some $n \in \mathbb{N}$ (Here we have used definition of finite set)

$\Rightarrow g \equiv f^{-1} : \mathbb{N} \to \mathbb{N}_n$ is also a bijection

In particular we consider, $h : \mathbb{N}_{n+1} \to \mathbb{N}_n$ defined as $h(x) = g(x) \quad \forall x \in \mathbb{N}_{n+1}$

Now we wish to prove h is a one-one function.

Let x_1 and x_2 be any elements of $\mathbb{N}_{n+1}$ such that

$$h(x_1) = h(x_2)$$
$$\Rightarrow g(x_1) = g(x_2)$$
$$\Rightarrow x_1 = x_2 \quad (\because g \text{ is } 1-1)$$
$$\Rightarrow h \text{ is } 1-1$$

which contradicts the Pigeonhole principle.

Thus our assumption is wrong i.e., the set $\mathbb{N}$ of natural numbers is an infinite set.

Hence proved.

Theorem: If A is a set with m elements and B is a set with n elements and if $A \cap B = \phi$, then $A \cup B$ has $m+n$ elements.

Proof:

Given that,

A is a set with m elements

$\Rightarrow \exists$ a bijection $f : \mathbb{N}_m \to A$ ...(i)

(Here we have used definition of finite set)

And

B is a set with n elements

$\Rightarrow \exists$ a bijection $g : \mathbb{N}_n \to B$ ...(ii)

To show there exists a bijection $h : \mathbb{N}_{m+n} \to A \cup B$.

We define h as

$$h(i) = \begin{cases} f(i), & i = 1, 2, \ldots, m \\ g(i-m), & i = m+1, m+2, \ldots, m+n \end{cases}$$

Claim: h is $1-1$

For $i \neq j$ we wish to show $h(i) \neq h(j)$.

Case 1: If $i, j \le m$ then $h(i) = f(i),\ h(j) = f(j)$.

Since f is $1-1$ $\Rightarrow f(i) \neq f(j) \Rightarrow h(i) \neq h(j)$ ... (iii)

Case 2: If $i, j > m$ then $h(i) = g(i-m),\ h(j) = g(j-m)$.

Since g is $1-1$ $\Rightarrow g(i-m) \neq g(j-m) \Rightarrow h(i) \neq h(j)$ ... (iv)

Case 3: If $i \le m,\ j > m$ then $h(i) = f(i),\ h(j) = g(j-m)$.

Since $f(i) \in A,\ g(j-m) \in B$ *and* $A \cap B = \phi$

$\Rightarrow f(i) \neq g(j-m) \Rightarrow h(i) \neq h(j)$ ... (v)

Combining (iii), (iv), (v) we get h is $1-1$ or injection.

Claim: h is onto

Let $x \in A \cup B \Rightarrow x \in A$ *or* $x \in B$.

If $x \in A \Rightarrow x = f(k)$ for some $k \in \mathbb{N}_m$

$\Rightarrow x = h(k)$

If $x \in B \Rightarrow x = g(k)$ for some $k \in \mathbb{N}_n$

$\Rightarrow x = h(m+k)$

Thus h is onto or surjection.

Since h is $1-1$ and onto $\Rightarrow$ h is a bijection.

Hence $A \cup B$ has $m+n$ elements.

Proved.

Theorem: If A is a set with $m \in \mathrm{N}$ elements and $C \subseteq A$ is a set with 1 element, then A \C is a set with m – 1 elements.

Proof: Home Work

Theorem: If C is an infinite set and B is a finite set, then C\B is an infinite set.

Proof: Home Work

Theorem: Suppose that S and T are sets and $T \subseteq S$.

(a) If S is a finite set, then T is a finite set.
(b) If T is an infinite set, then S is an infinite set.

Proof:

(a) Given that S is a finite set.

Suppose, $T = \phi$.

We know that empty set is finite.
$\Rightarrow$ T is a finite set.

Suppose that $T \neq \phi$.

We have to prove this theorem using principle of mathematical induction.

Let S has 1 element, then the only nonempty subset T of S must coincide with S, so T is a finite set.

Let us assume that every nonempty subset of a set with k elements is finite, i.e., if S has k elements and T is a nonempty subset of S then T is a finite set.

Now let S be a set having $k+1$ elements (so there exists a bijection f of N_{k+1} onto S), and let $T \subseteq S$.

If $f(k+1) \notin T$, we can consider T to be a subset of $S_1 = S \setminus \{f(k+1)\}$.

We know that If A is a set with $m \in \mathrm{N}$ elements and $C \subseteq A$ is a set with 1 element, then $A \setminus C$ is a set with $m - 1$ elements.

In view of the above statement we have S_1 has k elements.

Hence, by the induction hypothesis, T is a finite set.

On the other hand, if $f(k+1) \in T$, then $T_1 = T \setminus \{f(k+1)\}$ is a subset of S_1. Since S_1 has k elements, the induction hypothesis implies that T_1 is a finite set. But this implies that $T = T_1 \cup \{f(k+1)\}$ is also a finite set.

(b) We know that, by contrapositive implication, the implication $P \Rightarrow Q$ is logically equivalent to the implication $(not\, Q) \Rightarrow (not\, P)$.

In view of the above statement we have that if T is an infinite set, then S is an infinite set.

Hence proved.

References:

(i) Bartle, R.G. and Sherbert, D. R.(2002). *Introduction to Real Analysis*, 3rd Ed., John Wiley and Sons (Asia) Pvt. Ltd., Singapore.

(ii) Malik, S. C. and Arora, S. (2008). *Mathematical Analysis*, 3rd revised edition, New age International (P) Ltd. New Delhi.